What is fiscal sustainability? —Transversality condition, Domar condition, the fiscal theory of the price level—*

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Abstract

This paper reviews previous studies on fiscal sustainability and conducts a pilot empirical analysis for Japan. In Section I, we summarize the definition of fiscal sustainability and the argument over sovereign defaults. Section II outlines the transversality condition as a typical formulation of fiscal sustainability. Section III introduces money and examines the transversality condition of nominal values. In this setting, the fiscal theory of the price level (FTPL), which asserts that the transversality condition is satisfied by price fluctuations, becomes valid. Section IV discusses three issues, taking into account the comparison of interest rates and economic growth rates (i.e., the Domar condition). First, based on the Domar condition, we discuss the equivalence/non-equivalence between the transversality condition and the convergence of the government debt-to-GDP ratio. Second, we consider the Domar condition from the perspective of dynamic efficiency and its implications for fiscal sustainability. In particular, we suggest that in the case where $g > r$, a rational bubble can arise in government debt and that convergence of the government debt-to-GDP ratio below a certain value (rather than the transversality condition) is an appropriate sustainability indicator. Third, we conduct an empirical analysis (stochastic debt sustainability analysis: SDSA) when there is uncertainty in $r - g$. According to our calculations, when there is uncertainty in $r - g$, even if the current situation is $g > r$, Japan's government debt-to-GDP ratio could be very large in 2030.

Keywords: fiscal sustainability, transversality condition, fiscal theory of the price level (FTPL), seigniorage, Domar condition, rational bubble, stochastic debt sustainability analysis (SDSA)

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I. Introduction

I-1. What is fiscal sustainability?

The COVID-19 outbreak in 2020 forced governments around the world to undertake massive fiscal stimulus packages. This fiscal response has also given rise to concerns about fiscal sustainability. Although economists do not fully agree on the exact definition of fiscal sustainability, the European Commission has defined it as follows.[1]

> *"The sustainability of public finances also referred to as fiscal sustainability, is the ability of a government to sustain its current spending, tax and other-related policies in the long run without threatening its solvency or defaulting on some of its liabilities or promised expenditures."*

In other words, fiscal sustainability is the ability to maintain current spending and tax levels in the long run without causing default on government debt or changes to the planned social security system. It is worth noting that fiscal sustainability does not only mean non-default on government debt but also that public finances are managed stably. Even if a country can issue an unlimited number of government bonds in its own currency as argued by MMT (Modern Monetary Theory), this does not mean that it is sustainable.

According to economic theory, fiscal sustainability is diagnosed by either (1) the convergence of the discounted present value of government debt to zero or (2) the convergence of the ratio of government debt-to-GDP to a constant value. Condition (1) is known as the "transversality condition" and is a standard setting in economic theory (Sections II and III). On the other hand, condition (2) is commonly used by the IMF, OECD, and World Bank. As we will see in Section IV, there is a close relationship between conditions (1) and (2), which are identical depending on the state of the economy. Even if viewed as different conditions, both conditions are common in that government debt is judged to be unsustainable if it is increasing at a certain rate. Fiscal sustainability is an issue because if it is not sustainable, the government will be forced to change its fiscal plans (including default) in the future. With this approach, governments are examining the sustainability of their public finances and revising their fiscal plans to avoid the need for sudden adjustments.

I-2. Is this time different?

Almost a decade after the publication of Reinhart and Rogoff (2009), the title of this book ridiculed states facing a debt crisis for not facing up to the crisis by saying "this time it is different." This book covers a wide range of issues, including not only public debt but also external debt and financial crises, whereas Reinhart and Rogoff (2010), published a lit-

[1] European Commission (2016), "European Semester: Thematic factsheet - Sustainability of public finances - 2017."

tle later, focuses on the effects of public debt on economic growth. The collapse may manifest itself as a serious macroeconomic crisis, such as a decline in economic growth or high inflation, in addition to cases of government cash-flow problems. The paper provides empirical evidence that countries with government debt levels exceeding 90% of GDP have significantly lower economic growth rates.[2] Subsequently, Herndon, Ash and Pollin (2014) criticized the calculation process for the results of this empirical analysis, giving rise to much debate about the 90% threshold.[3]

Despite these controversies, Reinhart and Rogoff (2009) had the contribution of building a comprehensive worldwide database of fiscal crises. The Reinhart and Rogoff (2009) database contains 66 country data covering 800 years, but only since 1800 there have been more than 250 external debt defaults and more than 70 domestic debt defaults. Domestic debt defaults are rarely reported in international news unlike external debt and the documents are written in the respective national languages in many cases, hence few cases of domestic debt default were known before Reinhart and Rogoff (2009). In many countries (the United States is no exception), there is also a significant lack of transparency in government accounting. In summary, the greatest impact of Reinhart and Rogoff (2009) was in pointing out that defaults on domestic debt are occurring far more frequently than previously thought.

In addition, the BoC-BoE Sovereign Default Database jointly produced by the Bank of Canada and the Bank of England since 2014, contains examples of public debt defaults in 154 countries from 1960 to 2020. The database reports that 146 governments have defaulted since 1960 and that there are 32 default cases for domestic currency-denominated debt.[4]

I-3. Sovereign default

When considering sovereign default, the definition of default is the most important issue. Reinhart and Rogoff (2009) and the BoC-BoE Sovereign Default Database include not only defaults in the narrow sense of total or partial default on public debt, but also debt rescheduling (changes of interest rates and maturities), exchange of securities with unfavorable terms, and currency reforms such as exchange of central bank notes for new currency with unfavorable terms, such as total or partial default on public debt, as well as debt rescheduling (change of interest rate or maturity), exchange of debt for securities with unfavorable terms, and exchange of the current currency for a new currency with unfavorable terms. In the case of debt denominated in the national currency, the government can pay the debt by issuing money, and the resulting inflation can reduce the real burden of the debt. Hungary in 1945-46 and Zimbabwe in 2007-08 are prime examples. Such debt repayment through money issuance and inflation may theoretically be considered a de facto default but

[2] Later, Reinhart, Reinhart and Rogoff (2012) revised the magnitude of the growth rate decline.
[3] Recently, Bitar, Chakrabarti and Zeaiter (2018) is positive to Reinhart and Rogoff, and Amann and Middleditch (2020) is negative.
[4] In the Reinhart and Rogoff (2009) dataset, domestic debt includes debt denominated in foreign currencies. The BoC-BoE Sovereign Default Database, on the other hand, distinguishes between domestic currency-denominated and foreign currency-denominated debt.

is not included in explicit public debt defaults in these databases. The reason why defaults on local currency-denominated debt have been considered extremely rare in the past is that such inflationary repayments were thought to be possible.

However, as we have already seen, there are many cases of defaults on local currency-denominated debt. Why do governments sometimes default on their currency-denominated debt, even though it can be repaid through inflation? One reason, also cited by Reinhart and Rogoff (2009), is that inflation creates distortions in the banking system and financial sector so that if the distortions caused by inflation are greater than the cost of default, the government will (rationally) choose to default. In addition, high inflation often comes down to currency reform. If the currency reform is done on terms that are unfavorable to the currency (money) holders, then it will count as a default. For example, in 1946, after World War II, Japan imposed a deposit blockade with a limit of 300 yen per month for the head of the household and 100 yen per household member, and switched between the old yen and the new yen, for reasons of inflation control. This constituted currency reform under unfavorable conditions and is documented in Reinhart and Rogoff (2009) as a case of default. As will be discussed in more detail in Section III, the currency is a debt of the integrated government, so just as exchanging government bonds for securities on unfavorable terms is considered a default, exchanging currency for new currency on unfavorable terms is also considered a default.

Furthermore, as shown in Reinhart and Rogoff (2009), high inflation is often accompanied by exchange rate crashes (currency crises). Then there may be a need to defend the currency against an exchange rate collapse. For example, in the 1990s, Russia introduced a target exchange rate system (corridor) to control the depreciation of the ruble, which caused rapid inflation as well as the depreciation of the ruble, due in part to the confusion caused by the transition to a market economy and the expansion of the budget deficit. In this context, the ruble came under speculative attack in 1997, and although the Central Bank of Russia (CBR) defended its currency against it, it lost nearly $6 billion in foreign reserves.[5] Subsequent declines in the prices of crude oil and non-ferrous metals have hurt the earnings of Russia's key industries, increasing the pressure for capital outflows and the depreciation of the ruble. Then, in 1998, the ruble was attacked again and the country sought financial assistance from the IMF, but negotiations temporarily broke down, and investors, fearing default, fled the market by selling Russian government bonds and securities. As a result, the Russian government declared a devaluation of the ruble and a default on its own currency-denominated debt.

I-4. Structure of this paper

As discussed above, sovereign default can take various cases. In other words, when fiscal sustainability is not maintained, various situations can be expected, including (1) default,

[5] Chiodo and Owyang (2002).

(2) high inflation, and (3) currency crisis. The significance of analyzing fiscal sustainability is to consider fiscal management that can be sustained over the long run without causing these crises.

The purpose of this paper is to provide a theoretical perspective on the concept of fiscal sustainability and its implications for the Japanese economy. Section II explains the traditional concept of fiscal sustainability in the real economy without money. The theoretical key to fiscal sustainability is the transversality condition discussed at the beginning of this section. We can derive the concept of an intertemporal budget constraint of government from this transversality condition. Section III introduces money into the model and discusses fiscal sustainability when government debt can be repaid through money issuance. The Fiscal Theory of Price Level (FTPL) discusses fiscal sustainability under these circumstances. FTPL's view differs from the traditional view of the government's intertemporal budget constraint. Section IV considers fiscal sustainability when the economic growth rate is higher than the interest rate, which is currently the focus of most attention. We then conduct an empirical analysis of fiscal sustainability in Japan using a framework that takes into account uncertainty in the difference between the interest rate and the economic growth rate.

II. Fiscal sustainability and transversality condition

This section discusses fiscal sustainability in terms of real variables. In other words, money is ignored here and a central bank is not considered. A discussion of the case where money exists, i.e., in nominal terms taking prices and inflation into account, is presented in Section III.

II-1. Standard framework

First, let us consider the budget constraint of government in a single year. The subscripts denote the period, g is government spending excluding interest payments, τ is tax revenue, b is the outstanding government bonds (debt) at the end of the period, and r is the government bond interest rate measured in real terms. In this case, the government's budget constraint for period t can be written as

$$g_t + r_{t-1}b_{t-1} = \tau_t + (b_t - b_{t-1}). \tag{2.1}$$

where the second term on the left-hand side, $r_{t-1}b_{t-1}$, is the interest payment cost of government bonds issued up to period $t-1$, and the second term on the right-hand side, $b_t - b_{t-1}$, means the amount of new government bond issuance in period t. In the following, the real interest rate r is assumed to be constant over time $(r_t = r)$.

Solving equation (2.1) s periods into the future yields

$$(1+r)\,b_{t-1} = \frac{1}{(1+r)^{s}}\,b_{t+s} + \sum_{i=0}^{s} \frac{\tau_{t+i} - g_{t+i}}{(1+r)^{i}}. \tag{2.2}$$

Looking at each term of equation (2.2), the left-hand side is the outstanding government bonds (including interest payments) existing at the beginning of period t (end of period $t-1$), the first term on the right-hand side is the present value of outstanding government bonds (value in period t) existing at the end of period $t+s$, and the second term on the right-hand side is the sum of the present value of the primary surplus (difference between tax revenue and government spending) from period t to period $t+s$. Assuming that s is finite, it is clear that equation (2.2) can be thought of as a budget plan from period t to period $t+s$. The government has already issued b_{t-1} government bonds in period t. If the sum of the present value of the primary surplus is positive (the second term on the right-hand side is positive) for the future, then the present value of the remaining outstanding government bonds in the final year of the plan (period $t+s$) is that much smaller. If the sum of the present value of the primary surplus for the planning period is in deficit (the second term on the right-hand side is negative), the opposite occurs. In other words, the present value of the outstanding government bonds remaining in the final year of the plan (period $t+s$) will be larger by that amount. If the government were to terminate in the final year of the plan (period $t+s$), the following conditions would be imposed to avoid eventual insolvency,

$$\frac{1}{(1+r)^{s}}\,b_{t+s} \le 0. \tag{2.3}$$

To borrow an expression from Kamiya (2000), equation (2.3) is called the condition prohibiting leaving debt.

However, if a new government is created after the $t+s$ period, then the debt may be left in the $t+s$ period relying on the new government's budget plan. While the current government may not be permanent, let us assume that at least the government's budget plan is formulated with an infinite horizon. Equation (2.2) is rewritten as follows,

$$(1+r)\,b_{t-1} = \lim_{s\to\infty} \frac{1}{(1+r)^{s}}\,b_{t+s} + \sum_{i=0}^{\infty} \frac{\tau_{t+i} - g_{t+i}}{(1+r)^{i}} \tag{2.4}$$

where the condition prohibiting leaving debt is extended to the infinite horizon, we obtain

$$\lim_{s\to\infty} \frac{1}{(1+r)^{s}}\,b_{t+s} \le 0. \tag{2.5}$$

Borrowing again a definition from Kamiya (2000), equation (2.5) is called the No-Ponzi Game Condition. It is well known that No-Ponzi games are not sustainable in the long run.

II-2. Transversality condition and intertemporal budget constraint

One of the conditions for dynamic optimization is often imposed by

$$\lim_{s\to\infty}\frac{1}{(1+r)^s}b_{t+s}=0\,, \tag{2.6}$$

which is a special case of equation (2.5). Under equation (2.6), not only excess liabilities at present value but also excess assets at present value are not allowed. Equation (2.6) is called the *transversality condition* and is a common measure of fiscal sustainability based on dynamic macroeconomics.[6] It may be useful to note the difference between equation (2.6) and $\lim_{s\to\infty} b_{t+s} = 0$. In order for equation (2.6) to hold, $\lim_{s\to\infty} b_{t+s} = 0$ is not a necessary condition. If the rate of increase of b is small compared to the discount rate r, equation (2.6) holds even if $\lim_{s\to\infty} b_{t+s} \neq 0$.

Assuming that the transversality condition (2.6) holds, the government's budget plan for the future can be expressed in

$$(1+r)b_{t-1}=\sum_{i=0}^{\infty}\frac{\tau_{t+i}-g_{t+i}}{(1+r)^i}\,, \tag{2.7}$$

where government debt at the present time corresponds to the present value of the sum of future primary surplus. Equation (2.7) is called the Intertemporal Government Budget Constraint (IGBC) and is imposed a priori on the government sector in standard dynamic optimization models.

While the above transversality condition is from the government's perspective, the private sector is not unaffected. This is because if government bonds are issued domestically, it is the private sector that provides loans to the government. Due to the lending and borrowing relationship, when the government's transversality condition is not satisfied, the private sector's transversality condition is also not satisfied. Therefore, when discussing the transversality conditions of government, the behavior of the private sector should also be taken into account. The discussion, including the private sector, is discussed in detail in Section IV.

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[6] As Kamiya (2000) and Kamihigashi (2001, 2002, 2003) point out, the transversality condition is a sufficient but not always necessary condition for dynamic optimization. There can be optimal solutions that do not require a transversality condition. Nevertheless, following Kamihigashi (2001), the transversality condition is a necessary condition for optimization in many (but not all) cases in the standard macroeconomic setting.

III. The Fiscal Theory of the Price Level (FTPL)

III-1. Introducing money

In the previous section, we discussed only real variables, leaving monetary (or nominal) variables unaddressed. This section examines IGBC in the presence of money. The central bank issue money in many countries, and the fiscal authority and the central bank are interconnected through the transaction of government bonds and revenues and expenditures. In what follows, we present the budget constraints of the fiscal authority and the central bank, according to Walsh (2017).

The budget constraint of the fiscal authority in nominal terms for period t is given by[7]

$$G_t + i_{t-1}B^T_{t-1} = T_t + (B^T_t - B^T_{t-1}) + RCB_t \tag{3.1}$$

where G_t is nominal government spending, T_t is nominal tax revenue, B^T_t is the nominal total outstanding government bonds in period t, and i_t is the nominal interest rate on government bonds. $i_{t-1}B^T_{t-1}$ represents the nominal interest payment on the outstanding government bonds issued up to period $t-1$, and $B^T_t - B^T_{t-1}$ means new issues of nominal government bonds in period t. RCB_t is any direct receipts from the central bank.[8] The left-hand side of equation (3.1) represents the expenditures of the fiscal authority, whereas the right-hand side represents the revenues of the fiscal authority. The fiscal authority finances government spending and interest payments through tax revenues, government bond issues, and remittances from the central bank.

Next, the nominal budget constraint of the central bank for period t is given by,

$$(B^M_t - B^M_{t-1}) + RCB_t = i_{t-1}B^M_{t-1} + (M_t - M_{t-1}) \tag{3.2}$$

where B^M_t is the nominal outstanding government bonds held by the central bank, $B^M_t - B^M_{t-1}$ is the central bank's purchases of government bonds in period t, $i_{t-1}B^M_{t-1}$ is the central bank's receipt of interest payments from the fiscal authority. M_t is called the *monetary base,* or high-powered money, which represents the sum of the stock of currency held by the nonbank public and bank reserves. $M_t - M_{t-1}$ is the change of the monetary base in period t. From now on, we call M_t just "money" for simplicity. The left-hand side of equation (3.2) represents the expenditure of the central bank, while the right-hand side represents the revenue of the central bank. Equation (3.2) describes the behavior of the central bank, where the central bank issues non-interest-bearing base money, earns interest payments by

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[7] Note that the real variables in the previous section are represented in lowercase, whereas the nominal variables are represented in uppercase.

[8] In Japan, the profits of the Bank of Japan are legally required to be paid into the national treasury, except for the portion used for reserves and dividends to investors. Similar systems are in place at the Fed in the U.S. and other central banks in other countries.

purchasing government bonds, and pays their profits to the fiscal authority. Here, we assume that the central bank holds only government bonds as assets and does not hold stocks or bonds of the private sector. In addition, although many central banks have been paying an interest rate on reserves in recent years, we assume that there is no interest rate on central bank current accounts (the nominal interest rate is zero).

III-2. The consolidated government budget constraint

By integrating equations (3.1) and (3.2) and defining the outstanding government bonds held by the public as $B \equiv B_t^T - B_{t-1}^M$, we can obtain the following nominal consolidated government budget constraint:

$$G_t + i_{t-1}B_{t-1} = T_t + (B_t - B_{t-1}) + (M_t - M_{t-1}) \tag{3.3}$$

Equation (3.3) means that the consolidated government can finance its expenditures not only through tax and issuing government bonds but also through printing money.

Let P_t be the price level in period t, and divide equation (3.3) by P_t yields the real consolidated government budget constraint,

$$g_t + rb_{t-1} = \tau_t + (b_t - b_{t-1}) + \left(m_t - \frac{1}{1+\pi_t} m_{t-1}\right) \tag{3.4}$$

where, $g_t \equiv G_t/P_t$, $\tau_t \equiv T_t/P_t$, $b_t \equiv B_t/P_t$, $m_t \equiv M_t/P_t$, and π_t is the net inflation rate $(P_t/P_{t-1} - 1)$. $r \equiv \frac{1+i_{t-1}}{1+\pi_t} - 1$ represents the net real interest rate on government bonds, and as in the previous section, the real interest rate is assumed constant over time. Equation (3.4) indicates that the consolidated government must collect real goods by imposing a tax or by having the private sector hold government bonds or money for its real expenditures. Although the consolidated government can issue any amount of money without cost, it faces the same real resource constraint as households and firms.

III-3. Seigniorage

Generally, the revenue from money creation is called seigniorage. There are two definitions of seigniorage: (1) increase in money and (2) savings in nominal interest rates. In the following, we examine each case in nominal and real terms.

The first definition of seigniorage in nominal terms simply considers the increase in money in the last parenthesized term on the right-hand side of equation (3.3) $M_t - M_{t-1}$ as seigniorage. On the other hand, the last parenthesized term on the right-hand side of equation (3.4)

$$\frac{M_t - M_{t-1}}{P_t} = m_t - \frac{1}{1+\pi_t} m_{t-1},$$

represents real seigniorage in the first definition. In contrast to the nominal case, inflation π_t appears, and note that seigniorage is larger when inflation is positive. Real seigniorage in the absence of inflation ($\pi_t = 0$) is $m_t - m_{t-1}$, while in the presence of inflation, the seigniorage would be

$$m_t - \frac{1}{1+\pi_t} m_{t-1} = m_t - m_{t-1} + \frac{\pi_t}{1+\pi_t} m_{t-1}.$$

The above equation implies that inflation yields additional real revenue $\frac{\pi_t}{1+\pi_t} m_{t-1}$. Since $\frac{\pi_t}{1+\pi_t} \approx \pi_t$, the term $\frac{\pi_t}{1+\pi_t} m_{t-1}$ can be considered as tax revenue with a tax rate of π_t and a tax base of m_{t-1}. Hence, inflation can be interpreted as a tax on money, it is often called *inflation tax*.

The second definition of seigniorage considers seigniorage as the savings in nominal interest rates. The debt of the consolidated government is the sum of the public holdings of government bonds and money, so we define the nominal consolidated government debt as $D_t \equiv B_t + M_t$. Rewriting equation (3.3) using the definition of D_t, we obtain

$$G_t + i_{t-1}D_{t-1} = T_t + (D_t - D_{t-1}) + i_{t-1}M_{t-1}. \tag{3.5}$$

In equation (3.5), $i_{t-1}M_{t-1}$ appears on the right-hand side that represents the revenue side of the consolidated government. This means that the consolidated government must pay interest payments on government bonds but does not have to pay interest payments for money, so the savings in interest payments can be considered revenue for the government.

Similarly, by rewriting equation (3.4) by defining the real consolidated government debt as $d_t \equiv \frac{D_t}{P_t} = \frac{B_t + M_t}{P_t}$, we obtain the following real consolidated government budget constraint

$$g_t + rd_{t-1} = \tau_t + (d_t - d_{t-1}) + \frac{i_{t-1}}{1+\pi_t} m_{t-1}. \tag{3.6}$$

The last term $\frac{i_{t-1}}{1+\pi_t} m_{t-1}$ on the right-hand side of equation (3.6) represents real seigniorage. Since $\frac{i_{t-1}}{1+\pi_t}$ can be approximated as i_{t-1}, seigniorage in real terms is the savings in the nominal interest rate, as in the nominal case.

In summary, there are two definitions of seigniorage for the consolidated government: (1) the increase in money and (2) the savings in nominal interest rates. The difference between the two definitions stems from whether government debt is considered only government bonds held by the public or whether it is considered together with base money. However, as we will see in the next section, whichever definition is used, there is no difference in IGBC of the consolidated government.

III-4. Intertemporal Government Budget Constraint of the Consolidated government

Let s_{1t} be the real value of the first definition of seigniorage, and s_{2t} be the real value of the second definition of seigniorage, each of which can be expressed as follows:

$$s_{1t} \equiv m_t - \frac{1}{1+\pi_t} m_{t-1},$$
$$s_{2t} \equiv \frac{i_{t-1}}{1+\pi_t} m_{t-1}.$$

Using these definitions of seigniorage, equations (3.4) and (3.6) can be rewritten as follows:

$$g_t + rb_{t-1} = \tau_t + (b_t - b_{t-1}) + s_{1t}, \tag{3.7}$$
$$g_t + rd_{t-1} = \tau_t + (d_t - d_{t-1}) + s_{2t}. \tag{3.8}$$

Iterating equations (3.7) and (3.8) forward, we obtain

$$(1+r)b_{t-1} = \sum_{i=0}^{\infty} \frac{\tau_{t+i} - g_{t+i}}{(1+r)^i} + \sum_{i=0}^{\infty} \frac{s_{1t+i}}{(1+r)^i} + \lim_{i\to\infty} \frac{1}{(1+r)^i} b_{t+i}, \tag{3.9}$$
$$(1+r)d_{t-1} = \sum_{i=0}^{\infty} \frac{\tau_{t+i} - g_{t+i}}{(1+r)^i} + \sum_{i=0}^{\infty} \frac{s_{2t+i}}{(1+r)^i} + \lim_{i\to\infty} \frac{1}{(1+r)^i} d_{t+i}. \tag{3.10}$$

As shown in Buiter (2007), under the transversality condition for money

$$\lim_{i\to\infty} \frac{1}{(1+r)^i} m_{t+i} = 0, \tag{3.11}$$

the following relationship holds between the present values of the sum of the two definitions of seigniorage:

$$\sum_{i=0}^{\infty} \frac{s_{1t+i}}{(1+r)^i} = \sum_{i=0}^{\infty} \frac{s_{2t+i}}{(1+r)^i} - (1+r)m_{t-1}. \tag{3.12}$$

Using the relationship in equation (3.12), equation (3.9) can be written as

$$(1+r)\,d_{t-1} = \sum_{i=0}^{\infty} \frac{\tau_{t+i} - g_{t+i}}{(1+r)^i} + \sum_{i=0}^{\infty} \frac{s_{2t+i}}{(1+r)^i} + \lim_{i\to\infty} \frac{1}{(1+r)^i}\, b_{t+i},$$

and imposing the transversality condition for government bonds

$$\lim_{i\to\infty} \frac{1}{(1+r)^i}\, b_{t+i} = 0, \tag{3.13}$$

we obtain IGBC of the consolidated government

$$(1+r)\,d_{t-1} = \sum_{i=0}^{\infty} \frac{\tau_{t+i} - g_{t+i}}{(1+r)^i} + \sum_{i=0}^{\infty} \frac{s_{2t+i}}{(1+r)^i}. \tag{3.14}$$

Equation (3.14), which imposes the transversality conditions for money and government debt on equation (3.9), is equivalent to equation (3.10) with the transversality condition for total government debt

$$\lim_{i\to\infty} \frac{1}{(1+r)^i}\, d_{t+i} = 0. \tag{3.15}$$

The true transversality condition for households is equation (3.15), and here we assume that both government bonds and money are non-negative, so if equation (3.15) holds, then equations (3.11) and (3.13) also hold. The first term on the right-hand side of equation (3.14) represents the present value of the primary surplus, and the second term represents the present value of seigniorage (the savings in nominal interest rate). Equation (3.14) implies that the real value of existing government debt must equal the present value of the fiscal surplus, including seigniorage. IGBC for the consolidated government is that of IGBC in the absence of money in the previous section (2.7), with money included in government debt on the left-hand side and seigniorage added to the right-hand side.

III-5. The Fiscal Theory of the Price Level

Comparing equations (3.14) and (2.7), there is little difference between IGBC in the presence of money (3.14) and IGBC in the absence of money (2.7). The introduction of money, however, brings the possibility that the government's intertemporal budget constraint can be satisfied through the adjustment of the price level. This idea that IGBC is satisfied through the adjustment of the price level is known as the Fiscal Theory of the Price Level (FTPL). FTPL is originally advocated by Leeper (1991), Woodford (1994, 1995, 1996, 1998), Sims (1994), and Cochrane (1998, 2001, 2005), and comprehensive surveys are conducted by Canzoneri, Cumby, and Diba (2010), Leeper and Leith (2016), and Cochrane (2021). FTPL assumes that the real value of the outstanding government debt is not predetermined, while only the nominal value is predetermined. In other words, instead of $d_{t-1} = D_{t-1}/P_{t-1}$ being given at the beginning of period t, only D_{t-1} is given at the beginning of period t, and its real value D_{t-1}/P_t is endogenously determined by the price level in period t. For the sake of convenience in mathematical operations, let us redefine the nominal government debt as $A_{t-1} \equiv (1 + i_{t-1})B_{t-1} + M_{t-1}$, which includes interest payments. The real consolidated government budget constraint then becomes

$$\frac{1}{1+r}a_t = a_{t-1} + g_t - \tau_t - \frac{i_t}{1+i_t}m_t, \tag{3.16}$$

where $a_t \equiv A_t/P_{t+1}$. With the revision of the definition of government debt, the seigniorage term changes to $\frac{i_t}{1+i_t}m_t$ instead of $\frac{i_{t-1}}{1+\pi_t}m_{t-1}$, but the implication remains that it is the savings in nominal interest rates.

Define the fiscal surplus of the consolidated government, including seigniorage, as

$$sp_t \equiv \tau_{t+i} - g_{t+i} + \frac{i_t}{1+i_t}m_t, \tag{3.17}$$

Deriving IGBC of the consolidated government from equation (3.16), we obtain

$$\frac{A_{t-1}}{P_t} = \sum_{i=0}^{\infty}\frac{sp_{t+i}}{(1+r)^i}. \tag{3.18}$$

As mentioned earlier, $A_{t-1} = (1 + i_{t-1})B_{t-1} + M_{t-1}$ is predetermined at the beginning of period t, and let us further assume that the present value of the fiscal surplus $\sum_{i=0}^{\infty}\frac{sp_{t+i}}{(1+r)^i}$ is also given. FTPL expects that IGBC is satisfied by the adjustment of P_t in the denominator of the left-hand side of equation (3.18).

Following Cochrane (2021), we discuss an intuitive mechanism for the price level adjustment in FTPL through the example of a one-period economy. Suppose that the economy is at the final period T. On the morning of period T, the government redeems its debt with money, including interest payments, to debt holders. Since debt is inherited in nominal value, the government prints new money on the morning of period T, and debt holders receive $(1 + i_{T-1})B_{T-1}$ in money.

Thus, households hold $A_{T-1} = (1 + i_{T-1})B_{T-1} + M_{T-1}$ in money in the morning of period T. During the day in period T, households buy and sell goods with money, and the government spends $P_T g_T$ by printing more money. Finally, at night in period T, the government soaks up money by imposing tax $P_T\tau_T$. In this situation, unless

$$\frac{(1+i_{T-1})B_{T-1}+M_{T-1}}{P_T}=sp_T \tag{3.19}$$

is not satisfied, there must be an excess or shortage of money.

Equation (3.19) is a one-period version of equation (3.18). When the left-hand side of equation (3.19) is greater than the right-hand side, there is an excess of money issued over the amount needed to pay taxes, and people will seek to exchange more money for more goods. Then the price level would rise. Conversely, when the right-hand side of equation (3.19) is greater than the left-hand side, people have insufficient money to pay tax and will refrain from purchasing goods but rather will sell their goods to obtain money. In this case, the price level would fall. Through such mechanisms, the price level is adjusted to satisfy IGBC in FTPL.

Now, consider the implications of FTPL for fiscal sustainability. FTPL is the theory that the transversality condition is always satisfied by the adjustment of the price level, but as noted in Section I, fiscal sustainability does not refer only to explicit default by the government. A path cannot be considered sustainable if high inflation forces drastic fiscal or currency reform.

As Cochrane (2021) noted, unexpected inflation that devalues the real value of government debt can be considered substantial default. Whether an explicit default or substantial default occurs, a modification of the fiscal plan would be inevitable, so both are considered unsustainable from the perspective of fiscal sustainability.

IV. The Domar condition and fiscal sustainability

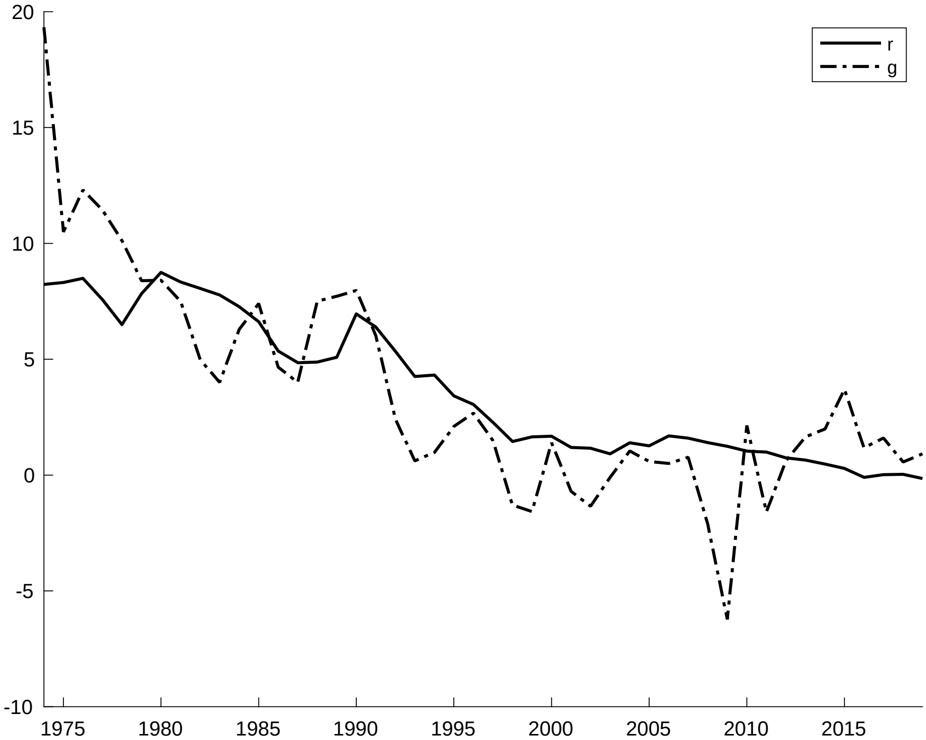
IV-1. Transversality condition and convergence of government debt-to-GDP ratios

In the previous sections, we have discussed fiscal sustainability based on the transversality condition, but in practice, fiscal sustainability is often defined as the convergence of the government debt-to-GDP ratio to a constant value. Theoretically, there is a strong relationship between this convergence condition and the transversality condition. Moreover, the positive or negative sign of the difference between the interest rate r and the economic growth rate g changes this relationship significantly. In Japan, the relationship between the interest rate r and the economic growth rate g is called the Domar condition, which has long been known to affect the convergence of the government debt-to-GDP ratio.[9]

It is helpful for the discussion to look at time series data on interest rates and economic growth rates before the theoretical analysis. Figure 1 shows time series data on government bond interest rates and GDP growth.[10] According to this figure, the economic growth rate was higher until about 1980, but there were many periods when the interest rate was higher

[9] The Domar condition is not the brainchild of Evsey D. Domar, despite its name. This condition was first proposed by two Japanese economists, Yonehara (1979) and Ara (1980), who extended the theorem of Domar (1944).

Figure 1. Government Bond Interest Rates and Economic Growth Rates (%)



Note: *g* is the growth rates of GDP; *r* is the government bond interest rates
Sources: Ministry of Finance, JGB Interest Rates. Cabinet Office, SNA.

for about 20 years after the 1990s. Recently, the interest rate has stabilized at a low level and the economic growth rate has exceeded the interest rate at times.

As we have seen above, both $r > g$ and $g > r$ can be real events in the real economy. Therefore, following Bartolini and Cottarelli (1994) and Escolano (2010), we organize the transversality condition and the (government debt-to-GDP) convergence condition separately for the $r > g$ and $g > r$ cases. In the following, as in Section II, we consider the case of real variables, expressing GDP in terms of Y_t and assuming that the economic growth rate is constant at g. Equation (2.4), the government's intertemporal budget constraint without the transversality condition, can be rewritten in terms of GDP ratios as

$$\frac{B_{t-1}}{Y_{t-1}} = \lim_{s\to\infty}\left(\frac{1+g}{1+r}\right)^s \frac{B_{t+s-1}}{Y_{t+s-1}} + \sum_{i=1}^{\infty}\left(\frac{1+g}{1+r}\right)^i \left(\frac{T_{t+i-1} - G_{t+i-1}}{Y_{t+i-1}}\right). \tag{4.1}$$

Thus the transversality condition when expressed as a GDP ratio is given by

[10] While *r* and *g* in the theoretical model discussed below are assumed to be real values, the government bond interest rates and GDP growth rates in Figure 1 are nominal values. However, this is not an issue when comparing the size of these two variables.

$$\lim_{s\to\infty}\left(\frac{1+g}{1+r}\right)^{s}\frac{B_{t+s-1}}{Y_{t+s-1}}=0. \tag{4.2}$$

Substituting equation (4.2) into equation (4.1) yields the government's intertemporal budget constraint (4.3) expressed as a ratio to GDP.

$$\frac{B_{t-1}}{Y_{t-1}}=\sum_{i=1}^{\infty}\left(\frac{1+g}{1+r}\right)^{i}\left(\frac{T_{t+i-1}-G_{t+i-1}}{Y_{t+i-1}}\right). \tag{4.3}$$

To start, consider the case where $r > g$. Since $1+r > 1+g$, $\lim_{s\to\infty}\left(\frac{1+g}{1+r}\right)^{s}$ is zero, equation (4.2) is satisfied if the government debt-to-GDP ratio (*B/Y*) converges to a constant value. It can also be explained as follows. Since we can approximate $\frac{1+g}{1+r}\approx 1+g-r$, the term $\left(\frac{1+g}{1+r}\right)^{s}$ decreases at the rate $r - g$ (> 0). Therefore, equation (4.2) is satisfied if the rate of increase in the government debt-to-GDP ratio is less than $r - g$. If $r > g$, the government debt-to-GDP ratio converges to a constant value, then the rate of increase of the government debt-to-GDP ratio is zero, and equation (4.2) is satisfied. This means that for $r > g$, convergence of the government debt-to-GDP ratio to a constant value is a sufficient condition for the transversality condition. It should be noted, however, that it is not a necessary condition. In other words, in the case $r > g$, the transversality condition may be satisfied even if the government debt-GDP ratio does not converge to a constant value. Thus, the (government debt-to-GDP ratio) convergence condition is a stronger condition than the transversality condition.

However, as Bohn (1991), Bartolini and Cottarelli (1994) and Escolano (2010) show, when there is a ceiling on the primary surplus-to-GDP ratio, a necessary condition for the transversality condition is that the government debt-to-GDP ratio does not exceed a defined range. Let us denote the primary surplus-to-GDP ratio as $(T_t - G_t)/Y_t \equiv pb_t$ and let the upper bound be $\overline{pb}$(> 0). If the government always operates within the primary surplus ceiling, then equation (4.3) can be written as

$$\overline{\left(\frac{B_{t-1}}{Y_{t-1}}\right)}=\frac{1+g}{r-g}\,\overline{pb}. \tag{4.4}$$

Equation (4.4) is the upper limit of the government debt-to-GDP ratio corresponding to the ceiling of the primary surplus. If the government debt-to-GDP ratio exceeds this level, the transversality condition is no longer satisfied. Thus, if there is an upper limit on the primary surplus-to-GDP ratio, then a necessary condition for the transversality condition is that the government debt-to-GDP ratio does not exceed a defined range. Thus, if $r > g$ and there is an upper bound on the primary surplus-to-GDP ratio, the transversality condition and the (government debt-to-GDP) convergence condition are equivalent.[11]

Next, consider the case $g > r$. It is clear from equation (4.2) that the transversality condition is not satisfied unless the government debt-to-GDP ratio is zero. In other words, even if the government debt-to-GDP ratio converges to a constant level, the transversality condition may still not be satisfied. Thus, the convergence condition for government debt-to-GDP ratio is a weaker condition than the transversality condition.

For example, consider a policy that produces a permanent constant primary deficit as a percentage of GDP ($pb < 0$). Equation (2.1) divided by Y_t and expressed as a GDP ratio yields

$$\frac{B_t}{Y_t} = \frac{1+r}{1+g}\frac{B_{t-1}}{Y_{t-1}} - pb. \tag{4.5}$$

Subtracting B_{t-1}/Y_{t-1} from both sides of equation (4.5) yields

$$\Delta\frac{B_t}{Y_t} = \frac{r-g}{1+g}\frac{B_{t-1}}{Y_{t-1}} - pb, \tag{4.6}$$

where $\Delta\frac{B_t}{Y_t} = \frac{B_t}{Y_t} - \frac{B_{t-1}}{Y_{t-1}}$. If $r > g$ and $pb < 0$, the right-hand side of equation (4.6) is always positive and the government debt-to-GDP ratio diverges, but if $g > r$, it always converges to a finite constant value B/Y such that

$$\frac{B}{Y} = \frac{1+g}{r-g}\,pb. \tag{4.7}$$

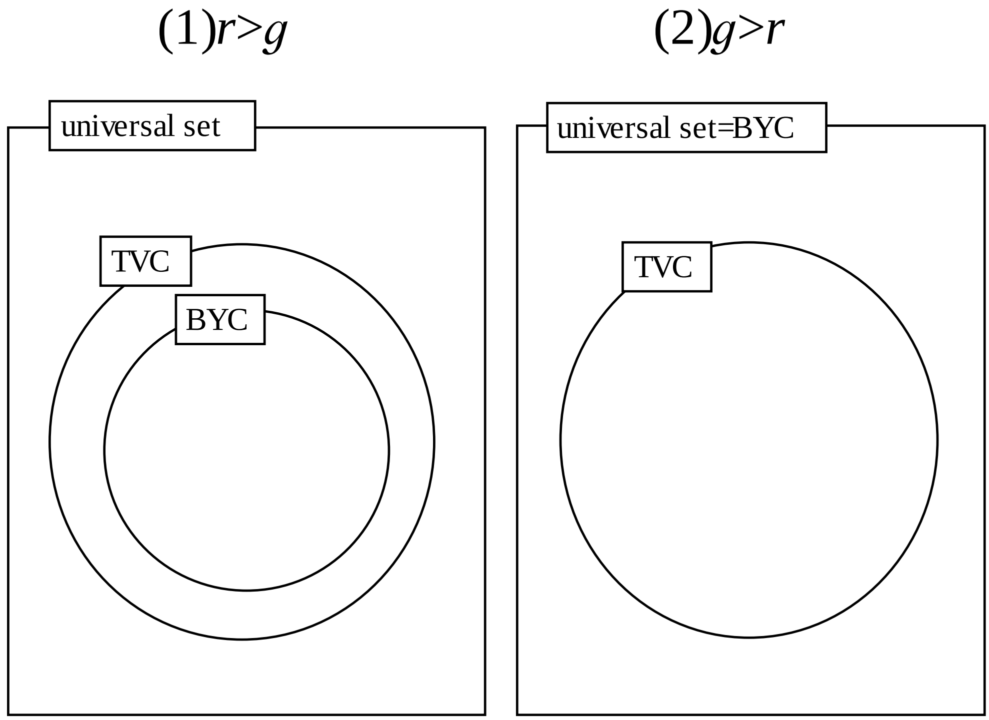
Thus, under $g > r$, the government debt-to-GDP ratio will always converge to a constant value. However, under $g > r$, even if the ratio of government debt-to-GDP converges to a constant value, the transversality condition in equation (4.2) is not satisfied unless it is zero. In other words, in the case $g > r$, the (government debt-to-GDP) convergence condition is a weaker condition than the transversality condition.

As described above, there is a close relationship between the (government debt-to-GDP) convergence condition and the transversality condition, and the nature of the relationship depends on the positive or negative sign of $r - g$. Let us summarize by noting the transversality condition as TVC and the convergence condition of the government debt-to-GDP ratio as BYC. Expressed in terms of necessary and sufficient conditions, the relationship between transversality and convergence conditions can be summarized by the sign $r - g$ as follows.

(1) $r > g$ ∶ BYC⇒TVC.

[11] Strictly speaking, as stated by Escolano (2010), there must be a lower limit as well as an upper limit to the government debt-to-GDP ratio. Since this paper assumes that government debt and GDP are always positive, this condition is implicitly satisfied.

Figure 2. The transversality condition (TVC) and the convergence condition (BYC)



(2) $g > r$: TVC⇒BYC.

In the case (1) $r > g$, BYC is a sufficient condition for TVC. However, as noted above, BYC is a necessary and sufficient condition for TVC if there is an upper limit on the primary surplus-to-GDP ratio. Also, in the case (2) $g > r$, TVC is a sufficient condition for BYC. However, note that BYC is always valid in the case $g > r$.

The same can be visually understood by representing it in a Venn diagram as shown in Figure 2. In the case of $r > g$, BYC is a subset of TVC, indicating that the convergence of the government debt-to-GDP ratio is a stronger condition than the transversality condition. In the case $g > r$, BYC is a universal set, and the convergence of the government debt-to-GDP ratio always holds, making the transversality condition a stronger condition.

IV-2. Dynamic efficiency and bubble

The relationship between the interest rate r and the growth rate g is also related to the condition of "dynamic efficiency" and the existence of a (rational) bubble. Let us assume the simplest neoclassical growth model. In the dynamic optimum, the following equation is satisfied:

$$f_K = \delta + \sigma(g - n), \tag{4.8}$$

where f_K is marginal product of capital, δ is the subjective discount rate of households, σ is the relative risk aversion (the inverse of the elasticity of substitution), and n is the population growth rate. Although there is no consensus on the value of δ or σ, Hatano and Yamada (2007), who use Japanese time series data, estimate σ at around 1. Rewriting equation (4.8) with $\sigma = 1$ yields

$$f_K = g + \delta - n\,. \tag{4.9}$$

Since the interest rate is equal to the value of f_K in a competitive market ($f_K = r$), the relationship between the interest rate and the economic growth rate g depends on the value of $\delta - n$. Again, according to Hatano and Yamada (2007), δ ranges from 0 to at most 0.2% per year. The Japanese population continues to decline, and n is about − 0.1 to − 0.2% in the last decade. In other words, although $\delta - n$ is slightly positive, the difference between r and g is negligible, thus we can consider that $r = g$.

Let us assume that $r = g$ holds in the optimal state as $f_K = r$. What happens when the economy deviates from the optimal state? When $r > g$, it is desirable to increase the savings for future consumption since $r > g$ implies a shortage of capital. In order to increase the savings rate, current consumption must be reduced, so a tradeoff arises between current consumption and future consumption. Under $r > g$, current and future consumption cannot be increased simultaneously, and there is no room for Pareto-improvement, so the state is Pareto-efficient.

Thus, the economy is called dynamically efficient when $r > g$ is satisfied. Conversely, $g > r$ means an excess of capital, and a lower savings rate is desirable. Lowering the savings rate means that current consumption can be increased. In other words, under $g > r$, current consumption and future consumption can be increased simultaneously, and there is room for Pareto-improvement (not Pareto-efficient). Therefore, the economy is called dynamically inefficient in the case of $g > r$.[12]

In the overlapping generations model, overaccumulation of capital may occur due to the imperfection of the capital market, resulting in $g > r$ in the steady state. In other words, the steady state may be dynamically inefficient in the overlapping generations model. Ihori (1978) shows that a permanent rollover of government debt is possible under such circumstances. According to Sakuragawa (2021), Tirole's (1985) "rational bubble" is essentially equivalent to Ihori's (1978) government debt.[13] A bubble is possible when the economy is dynamically inefficient. The rate of return on a rational bubble is the growth rate (higher than the interest rate) and crowd-out of capital occurs as people seek to purchase bubble assets rather than real capital. As a result, the marginal productivity of capital increases and dynamic efficiency ($r > g$) is achieved in the steady state. Thus, under $g > r$, government

[12] Note that "optimal" and "efficient" are different meanings. "Efficient" means being Pareto-efficient, while "optimum" is a state in which utility maximization is achieved. There are a number of possible Pareto-efficient states, and the optimal state is included among Pareto-efficient states.

[13] Sakuragawa (2021), p. 74.

debt act as a rational bubble and may lead to desirable outcomes.

As shown in Tirole (1985) and Sakuragawa (2021), the size of the bubble will be a constant share of GDP in a steady state. When government debt is considered a bubble asset, the government debt-to-GDP ratio will be constant. As discussed in IV-1, under $g > r$ (as well as under $g = r$), the transversality condition is not satisfied if the government debt-to-GDP ratio is constant above 0. Why is a positive government debt-to-GDP ratio possible under $g \geq r$ in the overlapping generations model? The reason is that the transversality condition given by equation (2.6) is an optimal condition for a representative household that lives for an infinite period of time, and equation (2.6) is not necessarily required for each individual in the overlapping generations model in which an infinite number of households are newly entering the economy. For example, consider Blanchard's (1985) perpetual youth model. In the perpetual youth model, the discount rate of households is not r but $r + \theta$, where θ is the mortality rate. Therefore, even if government debt grows at a rate greater than r, the present value of government debt for households will converge to zero if the rate is less than $r + \theta$. In other words, the transversality condition expressed in equations (2.6) or (4.2) need not be satisfied.

Even when households are assumed to live for an infinite period of time, if there are imperfections in the capital market, a bubble is possible. For example, in an economy where people face idiosyncratic risk regarding income, as represented by Aiyagari (1994), precautionary savings may lead to an overaccumulation of capital, which creates a bubble. Kocherlakota (1992), discussing the conditions for the existence of a bubble in an economy without capital, shows that a bubble arises when individuals face borrowing constraints and when the growth rate is higher than the interest rate.[14] Brunnermeier et al. (2020) introduce bubbles into FTPL. Iterating equation (3.16) forward, we obtain

$$\frac{A_{t-1}}{P_t} = \sum_{i=0}^{\infty} \frac{sp_{t+i}}{(1+r)^i} + \lim_{i \to \infty} \frac{1}{(1+r)^{i+1}} a_{t+i}. \tag{4.10}$$

Imposing the transversality condition for representative households

$$\lim_{i \to \infty} \frac{1}{(1+r)^{i+1}} a_{t+i} = 0, \tag{4.11}$$

on equation (4.10) yields the basic equation of FTPL (3.18). As noted above, the transversality condition for representative households is not necessarily satisfied under a bubble economy. Therefore, the second term on the right-hand side of equation (4.10) is not necessarily zero, and $\lim_{i \to \infty} \frac{1}{(1+r)^{i+1}} a_{t+i}$ becomes the bubble term. Then, Brunnermeier et al. (2020) state that equation (4.10) is not an equation that determines the price level, but the size of

[14] Kocherlakota (1992) also derives individual transversality conditions in the presence of borrowing constraints.

the bubble. If the government follows a rule that keeps the debt growth rate constant, or a fixed tax as in Bassetto and Cui (2018), the price level cannot be uniquely determined. Brunnermeier et al. (2020) show that the price level can be uniquely determined by adopting a state-dependent fiscal rule in which the government changes the growth rate of government debt depending on the level of government debt.

Finally, consider the case where $f_K = r$ does not hold. So far, a bubble is possible only in a dynamically inefficient economy. However, as Sakuragawa (2021) mentioned, $f_K = r$ does not necessarily hold when there is uncertainty or imperfection in the economy. For example, Blanchard (2019), Barro (2020), Ball and Mankiw (2021), Kocherlakota (2021), and Reis (2021) discuss the sustainability of government debt in a setting where the rate of return on capital is above the economic growth rate as

$$f_K \geq g > r \tag{4.12}$$

and dynamic efficiency is satisfied, while the interest rate on government bond r is below the economic growth rate. In such a case, a bubble is still possible when $g > r$, and thus a permanent rollover of government debt is also possible.

In summary, for $g > r$, fiscal sustainability may be satisfied as long as the government debt-to-GDP ratio converges to a certain value even if the transversality condition for representative households is not satisfied because a bubble is possible when $g > r$. As shown by Kocherlakota (1992) and Hirano and Yanagawa (2017), it should be noted that the transversality condition of each agent is satisfied in such an economy. In the case $g > r$, the government debt-to-GDP ratio may be a more appropriate criterion for fiscal sustainability. Bartolini and Cottarelli (1994) propose the convergence of the government debt-to-GDP ratio below a certain value as a milder criterion of fiscal sustainability when $g > r$.

However, Blanchard et al. (2021) and Blanchard (2021) point out two problems with assessing fiscal sustainability by the convergence of the government debt-to-GDP ratio under $g > r$. The first is the possibility that $r > g$ again at some point due to the increase in government debt causes a crowding out of real capital. The government would then need to run a primary surplus in order to converge the government debt-to-GDP ratio. As we saw in Section IV-1, fiscal policy is unsustainable if the government debt-to-GDP ratio is at a level that cannot be converged even by the upper value of the primary surplus. Second, although we have assumed that r and g are constant over time and there is no uncertainty, r and g are uncertain in practice. As shown in Figure 1, the relationship between the growth rate and the interest rate varies over time in Japan, and there is no guarantee that $g > r$ will always hold in the future. Thus, the actual decision on fiscal sustainability must consider the uncertainties in r and g.

IV-3. Stochastic Debt Sustainability Analysis (SDSA)

Blanchard et al. (2021) and Blanchard (2021) propose the Stochastic Debt Sustainability

Analysis (SDSA) as the recommended approach for examining fiscal sustainability in the presence of uncertainty in the difference between interest rate and growth rate. SDSA is an analysis similar to "The Deficit Gamble" presented in Ball, Elmendorf and Mankiw (1998), which calculates the distribution of the future government debt-to-GDP ratio under the presence of uncertainty in the difference between the interest rate and economic growth rate. In this section, we examine the fiscal sustainability of Japan using the simple SDSA model presented in Blanchard (2021).

The difference between the interest rate and the growth rate is assumed to be determined by the sum of the variable x_t which follows a random walk, and the white noise u_t;

$$(r_t - g_t) = x_t + u_t, \tag{4.13}$$
$$x_t = x_{t-1} + e_{xt}, \tag{4.14}$$
$$u_t = a_u + e_{ut}, \tag{4.15}$$
$$e_{xt} \sim N(0, s_x) \ i.i.d.$$
$$e_{ut} \sim N(0, s_u) \ i.i.d.$$

e_{xt} represents the long-run component of fluctuation in $r - g$ and e_{ut} represents the short-run component of fluctuation in $r - g$, both of which are assumed to follow a normal distribution. I.I.D. (independently and identically distributed) means that the random variables are independent and follow the same distribution. The distinction between short-run and long-run components of fluctuations in $r - g$ is the first difference from Ball, Elmendorf and Mankiw (1998) and Oguro (2009). a_u is the value of $r - g$ before the shock is realized at period 0. We assume that $a_u = 0$.

Equations (4.13)-(4.15) consist of a state-space model and parameters can be estimated by Kalman filter. As described in Kim and Nelson (1999), there are two ways to estimate parameters using the Kalman filter: (1) maximum likelihood estimation from the frequentist viewpoint, and (2) Bayesian estimation; this paper uses the former, maximum likelihood estimation. The procedure is as follows. First, given the initial values of the parameters, and the expectation and variance of the initial values of the state variables, the log-likelihood is calculated from the Kalman filter, and determines the value of the parameter that maximizes the log-likelihood. The expectation of the initial value of the state variable is set to 0. The expectations of the initial values of the state variables are set to 0. For the initial values of the variances, we adopt 10^7, which is the default value of the "dlm" package, one of popular packages to run the Kalman filter in the R language. The initial values of the parameters are set to $s_x = s_u = 1$.[15] To search for parameters to maximize the log-likelihood, we use the BFGS method (Broyden Fletcher Goldfarb Shanno algorithm), which is a kind of quasi-Newton method.

Table 1 shows the results of maximum likelihood estimation using Japanese annual data

[15] If the initial values of the parameters and the expectation and variance of initial values of the state variables are changed, the results of the estimation remaine almost the same.

Table 1: Estimated values of parameters

s_x	s_u
1.4907	0.7931

on the difference between the interest rate and the growth rate from 1975 to 2019. The interest rate data is based on the 9-year JGB nominal interest rate until 1985 and the 10-year JGB nominal interest rate after 1986. The growth rate data is constructed by simply connecting 68SNA and 2008SNA nominal GDP series.[16]

Uncertainty in $r - g$ affects the transition of the government debt-to-GDP ratio. Expressing equation (2.1) as a GDP ratio,

$$d_t = \frac{1+r_t}{1+g_t} d_{t-1} - pb_t, \tag{4.16}$$

where $d_t \equiv \frac{B_t}{Y_t}$, $pb_t \equiv \frac{T_t - G_t}{Y_t}$, and $1+g_t = \frac{Y_t}{Y_{t-1}}$. Equation (4.16) can also be viewed as allowing the interest rate and the growth rate to fluctuate in equation (4.5) discussed in Section IV-1. Since $r - g$ is determined from equation (4.13), we can now calculate the transition of d by giving the primary surplus pb. Ball, Elmendorf and Mankiw (1998) and Oguro (2009) treat the primary surplus as a deterministic variable, while the SDSA model considers the primary surplus as also having uncertainty. Blanchard (2021) proposes the following fiscal rule in which the primary surplus responds to the product of the government debt-to-GDP ratio and $r - g$

$$pb_t = a_s + c\,[(r_t - g_t)\,d_{t-1}] + e_{st}. \tag{4.17}$$
$$e_{st} \sim N(0, s_s)\ i.i.d.$$

Equation (4.17) is similar to the fiscal rule presented in Bohn (1998), except that $r - g$ is multiplied by the government debt-to-GDP ratio. The presence of $r - g$ allows the fiscal rule conditional on the difference between the interest rate and the growth rate. For example, when the growth rate is higher than the interest rate ($r - g < 0$), the government runs a primary deficit because fiscal sustainability is less likely to be a problem. When the interest rate is higher than the growth rate ($r - g > 0$), the government runs a primary surplus because fiscal sustainability is more likely to be a problem. Blanchard (2021) sets $c = 0$ for the benchmark case and examines how much the value of c affects fiscal sustainability.

Using the SDSA model given by equations (4.13)-(4.17), we calculate the distribution of the future government debt-to-GDP ratio in Japan. However, we consider the fiscal sustain-

[16] Specifically, we connect 68SNA and 2008SNA by calculating the ratios of the data during the overlapping period (1994-1998) and then multiplying the average of the values by the pre-1993 data of 68SNA.

Table 2. Parameter values used for simulation

a_s	s_x	s_u	s_s
-2.7498	1.4907	0.7931	3.2580

ability of government bonds only, ignoring the central bank and money. As in Blanchard (2021), the simulation period is 10 years, from 2020 to 2030. The values of the parameters used in the simulations are summarized in Table 2. The values of s_x and s_u are the estimated values by the Kalman filter in Table 1. For the benchmark case with $c = 0$, a_s is the mean of the primary surplus-to-GDP ratio and s_s is the standard deviation of the primary surplus-to-GDP ratio. Therefore, we calculate the mean and standard deviation of the primary surplus-to-GDP ratio from 1975 to 2019 in annual data, which we use as the a_s and s_s values. In constructing the series of primary surplus, payments and receipts of "property income" are added and subtracted, respectively, from the "general government net borrowing/net lending (deficit/surplus)" in the national accounts. The initial value of government debt is set d_0 = 176.6(%) based on Japan's outstanding government bonds-to-GDP ratio in 2020. The initial value of the state variable x is -0.62, the filtered value of x in 2019 by the Kalman filter.[17]

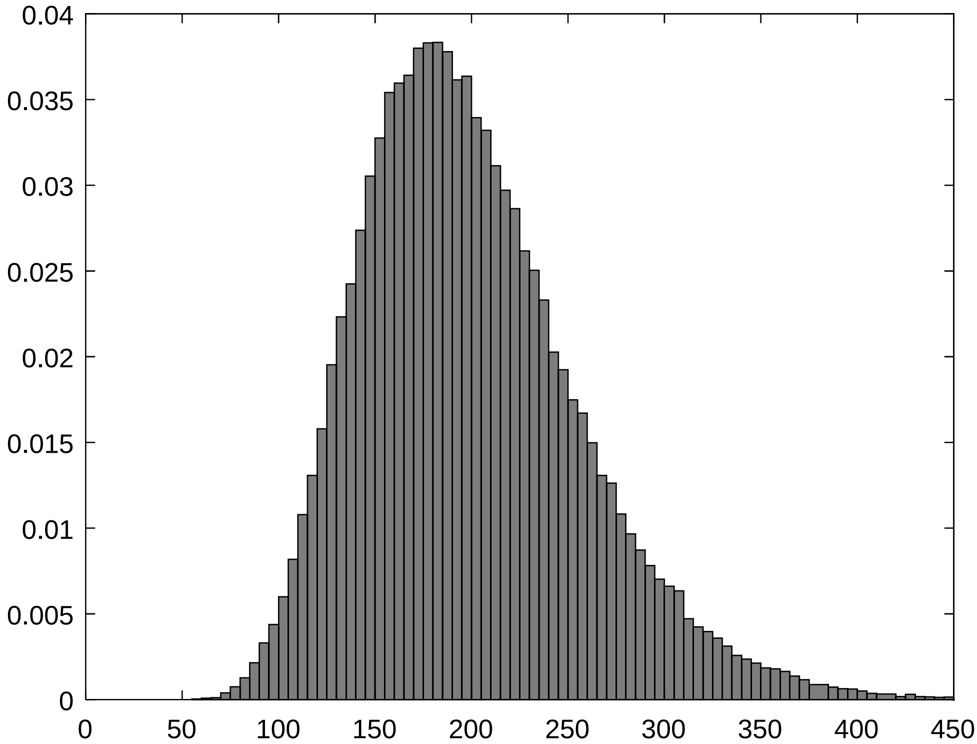
From the above setup, we generate a series of shocks using the Monte Carlo method and simulate the path of government bonds from 2020 to 2030 for 100,000 times. Figure 3 shows the distribution of the government debt-to-GDP ratio in 2030 in the benchmark case $c = 0$. For $c = 0$, the probabilities of the government debt-to-GDP ratio exceeding 200%, 250%, and 300% in 2030 are 44.4%, 17.35%, and 5.4%, respectively. The reasons for the asymmetry in the distribution can be considered as follows. The transition equation for the government debt-to-GDP ratio in equation (4.16) is a time-varying coefficient AR (1) model. Canova (1993) shows that in such a time-varying AR or VAR model, the (unconditional) distribution of the dependent variable after a one period ahead is not normal and can be asymmetric. Such a characteristic also appears here, but we will leave a detailed discussion for a future task. The asymmetry of these distributions seems to appear here, but we would like to leave a detailed study for a future task.

In SDSA, some threshold value is set for the government debt-to-GDP ratio, and if the probability of exceeding the threshold value is low (e.g., 5% or less), fiscal sustainability is considered to be guaranteed; otherwise, some fiscal adjustment is considered necessary. However, since it is difficult to determine what value should be used as the threshold, this paper does not provide a detailed discussion regarding the setting of the threshold. What is important here is that even with $g > r$ as a starting point, if there is uncertainty in $r - g$, the government debt-to-GDP ratio in 2030 may be quite large.

Table 3 shows the probabilities of the government debt-to-GDP ratio exceeding 200%, 250%, and 300% in 2030 for the same simulations with $c = 0.2$ and $c = 0.5$, respectively. As

[17] Since the SDSA model in this paper is a local-level model, the filtered values of 2019's state variables in 2019 and predicted values of 2020's state variables in 2019 are equal.

Figure 3. Distribution of the government debt-to-GDP ratio in 2030



Blanchard (2021) shows, an increase in c reduces the variance of the future government debt-to-GDP ratio, thus avoiding extreme increases such as 250% or 300%. In the Japanese data, a fiscal rule conditioned on the difference between the interest rate and the growth rate, as given by equation (4.17), would be effective in ensuring fiscal sustainability. However, since the SDSA model in this paper abstracts from the effects of fiscal expansion or austerity on aggregate demand, it is necessary to use a macroeconomic model to conduct an analysis that takes such endogenous changes in r and g into account.

There are still few empirical studies of fiscal sustainability in a $g > r$ environment. Mehrotra and Sergeyev (2020) find that the fiscal limit for the government debt-to-GDP ratio in the US is 150-220% based on a model with $g > r$. Lian, Presbitero and Wiriadinata (2020) find that high government debt leads to upward pressure on $r - g$ using data from advanced and emerging economies. Empirical analysis based on models with endogenous changes in interest rates and growth rates would be the subject of future work.

Table 3. Probability of different fiscal rules

	c=0	c=0.2	c=0.5
>200%	44.4%	45.2%	47.2%
>250%	17.3%	13.2%	5.4%
>300%	5.4%	2.5%	0.1%

V. Concluding remarks

This paper summarizes the debate on fiscal sustainability by reviewing previous studies up to recent years. Section I presents the basic concept of fiscal sustainability and a theoretical arrangement of sovereign defaults. Section II introduces the transversality condition in real terms as the most representative formulation of fiscal sustainability and derives the government's intertemporal budget constraint. Section III extended the discussion to nominal and introduced money and central banking explicitly. The key in this case is prices, and we outline the FTPL as a model in which prices are adjusted for fiscal sustainability. In Section IV, we discussed the convergence condition for the government debt-to-GDP ratio as another definition of fiscal sustainability. Although the convergence condition is commonly used, the relationship between the convergence condition and the transversality condition has not been discussed. In particular, it is important to organize the relationship between the convergence and the transversality conditions in a situation when the economic growth rate exceeds the government bond interest rate ($g > r$).

According to the discussion in Section IV-1, the relationship between the transversality condition and the convergence condition depends on the positive and negative sign of the difference between the interest rate r and the economic growth rate g, i.e., the dormer condition. In the case $r > g$, the convergence of the government debt-to-GDP ratio is generally a sufficient condition for the transversality condition. In this case, the convergence of the government debt-to-GDP ratio satisfies the transversality condition, but the transversality condition can be satisfied even if the government debt-to-GDP ratio does not converge. Thus, when $r > g$, convergence of the government debt-to-GDP ratio is a relatively stronger condition. Conversely, in the case $g > r$, the transversality condition is a sufficient condition for convergence of the government debt-to-GDP ratio. Therefore, in this case, the government debt-to-GDP ratio converges if the transversality condition is satisfied. In other words, the transversality condition is relatively stronger. Moreover, the government debt-to-GDP ratio always converges to a constant value in the case $g > r$.

Section IV-2 discussed the relationship between the interest rate and the economic growth rate from the perspective of dynamic optimization and dynamic efficiency. The key argument here is the relationship between the bubbles in the $g > r$ economy and fiscal sustainability. Since $g > r$ may be realized in the overlapping generations model, there may be a rational bubble in public finances as shown by Ihori (1978). It is important to note that in such a situation, the transversality condition for representative individuals is not satisfied, but the transversality condition for each of them is satisfied. Given this possibility, we should discuss the convergence of the government debt-to-GDP ratio below a certain value as an indicator of fiscal sustainability when $g > r$.

However, as Blanchard et al. (2021) and Blanchard (2021) point out, there is a problem with assessing fiscal sustainability by the convergence of the government debt-to-GDP ratio under $g > r$. The problem is that there is uncertainty in the value of $r - g$. Although there has not yet been enough empirical research that takes into account the possibility of $g > r$

and the uncertainty in the value of $r - g$, we piloted a Stochastic Debt Sustainability Analysis (SDSA) based on Blanchard et al. (2021) and Blanchard (2021) in Section IV-3. The conclusion of the empirical study was that the government debt-to-GDP ratio in Japan can be very large when there is uncertainty in $r - g$. Of course, since the results are affected by various parameter settings, further empirical studies, including verification of robustness, are needed.

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