Estimating the Yield Curve Using the Nelson-Siegel Model: Evidence from Daily Yield Data*

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Abstract

Since 2000, Japanese government bond yields have been very low. In this study, I estimate the yield curve using the Nelson-Siegel model with the daily yield data in Japan. This estimation focuses on the changes in the decay factor and examines how the estimates of the level, slope and curvature factors change. Based on the Nelson-Siegel model, I conduct the estimation using two methods—a linear estimation method under which the decay factor remains fixed, and a non-linear estimation method under which the factor is estimated every period. I find that the latter method achieves much a more precise estimation. This finding is prominent particularly under a low-interest-rate environment like the one that was observed in the 2010s. In addition, the estimation shows that the decay factor started decreasing gradually from the day when the BOJ decided to introduce Quantitative and Qualitative Monetary Easing and that the level factor fell steeply immediately after the decision to introduce the negative interest rate policy. Moreover, the volatility of the decay factor decreased after the introduction of yield curve control.

Keywords: zero-coupon bond, yield curve, Nelson-Siegel model, non-linear estimation
JEL Classification: E43, E44, E47

I. Introduction

Japanese government bond (JGB) yields have been low since the 1990s. The short-term JGB yields have been negative, and those with maturity of 10 years have been below 1 percent since 2000. The Bank of Japan (BOJ) introduced Quantitative and Qualitative Monetary Easing (QQE) in April 2013, and the negative interest rate policy and yield curve control in January and September 2016, respectively. To understand the low-interest-rate environment in Japan, Koeda and Sekine (2021) analyzed yield curves employing the Nelson-Siegel (1987; hereafter NS) model with the monthly yield data. They show that the decay factor in the NS model has declined substantially, and the decline pushed down the long-term yields and the conditional variance.

This study investigates the low-interest-rate environment in Japan employing the NS

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model with the daily yield data. In this study, I focus on changes in the decay factor, and analyze how much estimates of yield curve factors have changed over time. The recent related literature estimated yield curves employing the state-space model (Koeda and Sekine 2021; Shiratsuka 2021). The limitation is that the transition equation in the state-space model is VAR(1), and the frequency of the data used is monthly. Instead, in this study, I use daily yield data and estimate the NS model with nonlinear ordinary least squares (OLS) every period. The merit is that increasing the frequency of the data leads to understanding the daily changes in the yield curve factors, although the changes may not be smooth because the estimated factors are not time-varying. Thus, I can investigate how many days the yield curve factors changed after the monetary policy of the BOJ.

My results are as follows. First, I estimate the two NS models: the NS model with the fixed decay factor and the NS model under which I estimate the decay factor every period. As a result, I found that the latter model obtains a more precise estimation every period from April 3, 1989, to December 31, 2019, in terms of root mean squared error (RMSE). Especially, in the low-interest-rate environment like the one that was observed in the 2010s, estimating the decay factor every period is positively recommended. Second, focusing on monetary policy by the BOJ, I show that the decay factor has declined since the introduction of QQE. Also, the negative interest rate policy may lead to a substantial decrease in the level.

The structure of this paper is as follows. In Section II, I introduce the NS model. In Section III, I estimate and compare the two NS models. Also, I investigate how much the yield curve factors have changed over time. In Section IV, I conclude. In the appendix, I estimate the Svensson model.

II. Estimation

II-1. Methods

In this study, I estimate the NS model. The NS model is

$$y(\tau) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \beta_2 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right),$$

where $y(\tau)$ is yield with maturity of τ-month, and β_0, β_1, and β_2 are level, slope, and curvature in the NS model, respectively. Here, λ is the decay factor.

There are many papers on estimating the NS model (e.g., Diebold and Li 2006; Diebold, Rudebusch, and Aruoba 2006). Diebold, Rudebusch, and Aruoba (2006) estimate the NS model employing the state-space model with the one-step method. The observation equation is the NS model, and the transition equation is VAR(1) with a vector of the yield curve factors. In their model, one decay factor is estimated. Koeda and Sekine (2021) estimate the state-space model with a one-step method employing the extended Kalman filter.[1, 2] In their estimation, the estimated decay factor is time-varying. On the other hand, as for two-step estimation, the NS model is estimated every period with OLS or nonlinear OLS, and afterwards the VAR(1) model with the vector of the estimated yield curve factors is estimated with OLS. It is known that a one-step estimation can obtain a more precise estimation than a two-step estimation in terms of estimation accuracy.

However, the limitation of estimating the state-space model is that the frequency of the data used is monthly or quarterly because the form of the transition equation in the model is VAR(1). In this study, I estimate the NS model with OLS or nonlinear OLS using daily yield data instead of the state-space model. Here, I call the NS model with a fixed decay factor "NS-LS," and the NS model under which I estimate a decay factor every period "NS-NL." I compare the fit of NS-LS with that of NS-NL, and investigate how the estimated yield curve factors have changed over time.

In this study, I follow the following steps. First, employing Diebold and Li's (2006) state-space representation, I estimate the NS model with monthly yield data. In this estimation, I obtain one estimated decay factor. Next, I estimate NS-LS by OLS with the decay factor using daily yield data. Lastly, I estimate NS-NL by nonlinear OLS using the estimated yield curve factors and decay factor as initial values.

II-2. Data

In this study, I use daily zero-coupon JGB yields. The source of the data is Bloomberg. The maturity is 3, 6, 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 180, 240, and 360 months. The data span ranges from April 3, 1989, to December 31, 2019.

Figure 1 shows evolution of the daily JGB yields with maturities of 1 year, 5 years, and 10 years, respectively. These yields are over 8 percent at the beginning of the 1990s. However, the 1-year bond yield is nearly zero since the latter of the 2010s. The 10-year bond yield ranges from 0 to 2 percent, and almost zero since the latter of the 2010s.

III. Results

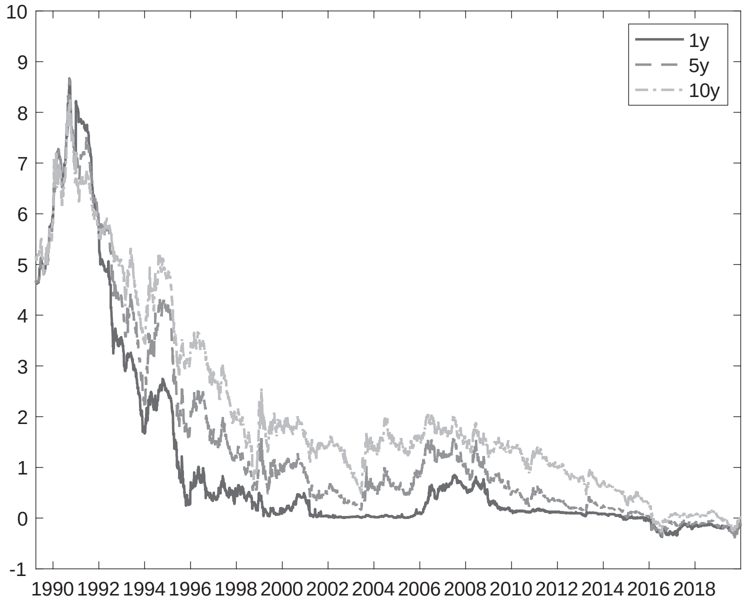
III-1. Estimation Results

First, I estimated the decay factor with monthly yield data employing the state-space model. The estimation period is from April 1989 to December 2019. To create the monthly data, I use the yields data of the end of the month. The estimated decay factor is 0.0247.

Next, I estimated the NS model by OLS with the fixed decay factor obtained from the state-space model (NS-LS). From the NS-LS, I obtained estimates of level, slope, and curvature. Lastly, using the decay factor and the estimates of level, slope and curvature obtained from NS-LS as initial values, I estimated the NS model by nonlinear OLS (NS-NL).

[1] The motivation of this paper depends on Koeda and Sekine (2021). The difference between their paper and this paper is that I estimate the NS model with daily yield data. Also, the estimation method in this study is nonlinear OLS. Moreover, I compare the fit from the nonlinear OLS with that from OLS by conducting the F-test.
[2] Koopman et al. (2010) estimate time-varying decay factors employing the GARCH model.

Figure 1. Evolution of daily yield data



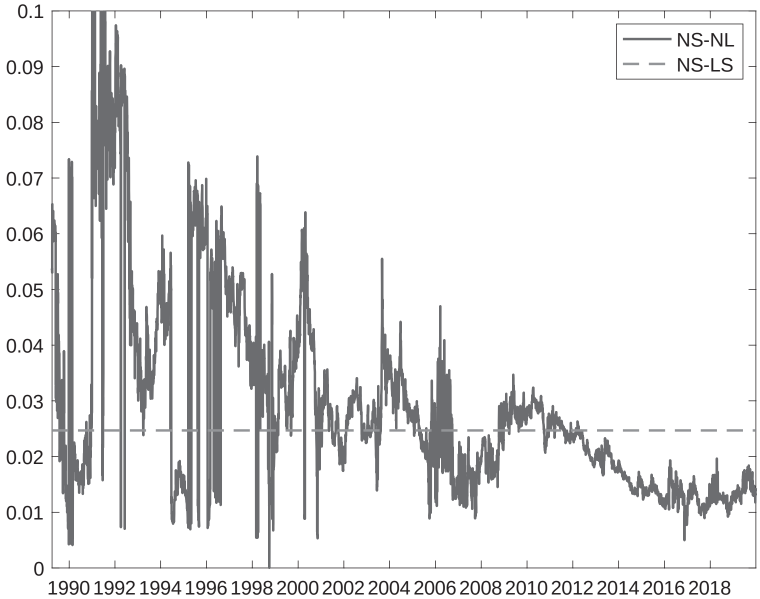
Note: This figure shows evolution of daily yield data in Japan. The straight, dashed, and dashed dotted lines indicate zero-coupon JGB yields with maturities of 1 year, 5 years, and 10 years, respectively. The horizontal line is date, and the vertical line is yields in terms of the annualized rate in percent.

From the NS-NL, I obtained not only estimates of level, slope, and curvature but also the estimate of the decay factor. The estimated decay factor is obtained every period in NS-NL.

Figure 2 shows evolution of daily estimates of the decay factor from NS-NL. In this figure, the decay factor was over 0.08 at the beginning of the 1990s, and afterwards, it decreased to 0.019 in 2019. The mean of the decay factor from April 3, 1989, to December 31, 1999, from January 3, 2000, to December 31, 2009, and from January 1, 2010, to December 31, 2019, is 0.045, 0.029, and 0.018, respectively. The decay factor of NS-NL is below that of NS-LS in the 2010s, and between 0.01 and 0.02 in the latter of the 2010s.

Figure 3 shows estimates of level, slope, and curvature from the two NS models. Panels a) and b) in the figure show the estimates obtained from NS-LS and NS-NL, respectively. In both models, the estimated level is positive and has decreased over time. The estimates of slope and curvature are both negatively estimated since the half of the 1990s. Focusing on estimates of NS-NL, curvature fluctuated largely in the half of the 1990s. In this period, the decay factor also fluctuated largely as shown in Figure 2. Focusing on level and slope, these estimates have moved in the opposite direction of each other since the latter of the 1990s. This result is consistent in that the sum of the level and slope in the NS model is the short rate.

Figure 2. Evolution of estimated decay factor



Note: This figure shows the decay factor estimated from NS-NL (straight line). The dashed line indicates the decay factor fixed in NS-LS (=0.0247).

III-2. Comparison of Models

Figure 4 shows RMSE computed from estimates of NS-LS and NS-NL, respectively. In the figure, RMSE is smaller in NS-NL among all periods of 8,021 periods. This result indicates that the fit of NS-NL is better in all periods.

Table 1 compares fits of NS-LS and NS-NL by conducting the F-test. In this study, the decay factor is fixed at 0.0247 in NS-LS. Thus, the null hypothesis is λ=0.0247. If this hypothesis is rejected, nonlinear OLS is better than OLS with the fixed decay factor in the period. The table shows the rejection rate of the null hypothesis with 1 percent, 5 percent, and

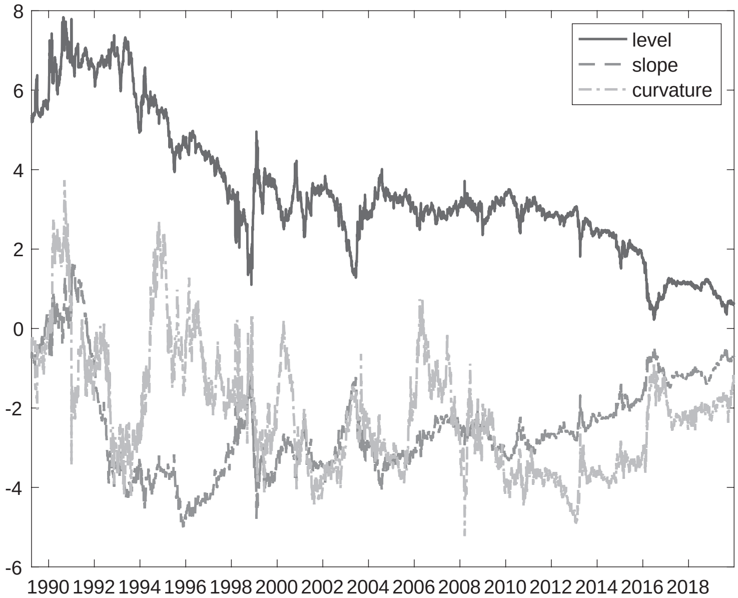
Table 1. Evaluation of the two NS models by F-test

	1989-1999	2000-2009	2010-2019	1989-2019
1% significance level	0.276	0.163	0.585	0.340
5% significance level	0.453	0.296	0.707	0.485
10% significance level	0.553	0.377	0.751	0.560

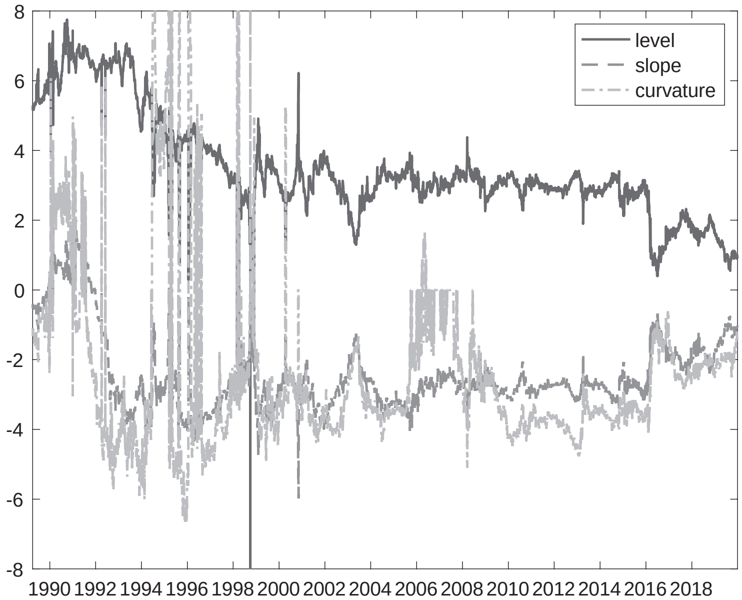
Note: This table shows the results of the rejection ratio by the F-test to compare the fit of NS-LS and NS-NL. The null hypothesis is λ=0.0247 because the decay factor is fixed at 0.0247 in NS-LS. Here, 1989-1999, 2000-2009, and 2010-2019 indicate April 3, 1989, to December 31, 1999, January 3, 2000, to December 31, 2009, and January 1, 2010, to December 31, 2019, respectively.

Figure 3. Evolution of estimates of level, slope, and curvature

a) NS-LS

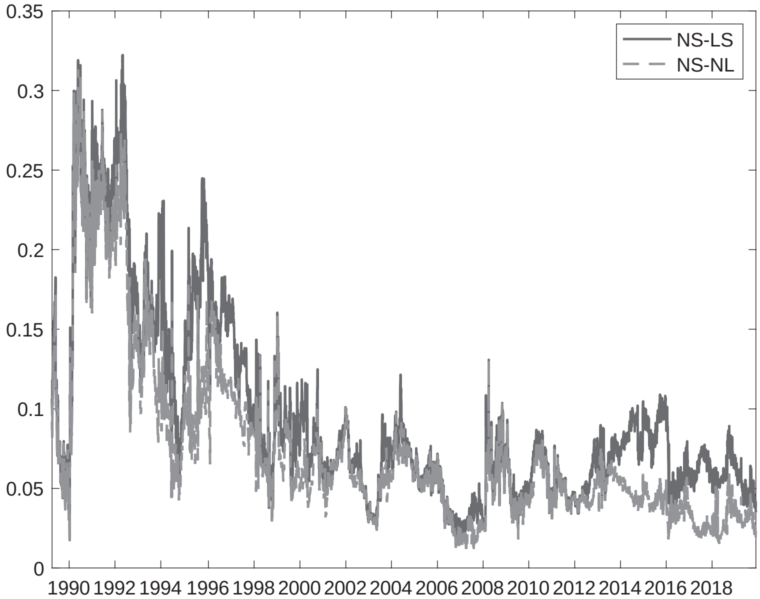


b) NS-NL



Note: This figure shows estimates of level, slope, and curvature from the NS model. Panel a) indicates those from NS-LS, and panel b) corresponds to those from NS-NL.

Figure 4. Comparison of RMSE



Note: This figure shows RMSE computed from the NS models. The straight and dashed lines indicate RMSE from NS-LS and NS-NL, respectively.

10 percent significance levels. In the table, the rejection rate is the highest in the 2010s, and is over 70 percent with the 5 percent significance level. On the other hand, the rate is the lowest in the 2000s, and is below 30 percent with the 5 percent significance level. In Figure 2, the mean estimates of the decay factor of NS-NL in the 1990s, 2000s, 2010s are 0.045, 0.026, 0.018, respectively. Since the decay factor of the null hypothesis is 0.0247 which is near 0.026, the rejection rate is relatively low in the 2000s. The rejection rate is relatively high in the 1990s and 2010s because the decay factors are far away from 0.0247. The effects of change in the decay factor tend to be larger when the decay factor is low. The rejection rate is around 50 percent with the 5 percent significant level in all periods. This indicates that nonlinear OLS is recommended as an estimation method especially in the period when the decay factor is very low.

Figure 5 shows the actual 10-year bond yields and the fitted yields from estimates of NS-LS and NS-NL in the 2010s. The period is the one in which NS-NL is the better model than NS-LS in terms of the small decay factor. The light straight and dashed lines indicate the fitted 10-year bond yields obtained from NS-LS and NS-NL respectively. The dark straight line corresponds to the actual yields. In this figure, there is little difference in the fitted values between NS-LS and NS-NL from 2010 to 2012 (782 periods in total). The mean decay factor of the three years is 0.025, and is nearly equal to the fixed decay factor of NS-LS. On the other hand, there is a difference between the actual and fitted yields of NS-LS since 2013. The mean decay factor between 2013 and 2019 (1,826 periods in total) is 0.014. The BOJ introduced QQE, negative interest rate policy, and yield curve control in April

Figure 5. Comparison of fit of 10-year bond yields

Note: This figure compares the actual yields with the fitted yields obtained from NS-LS and NS-NL since 2010. The light straight and dashed lines indicate those from NS-LS and NS-NL, respectively, whereas the dark straight line corresponds to the actual yields.

2013, January 2016, and September 2016, respectively. As shown in the next subsection, the monetary policy of the BOJ seems to decrease the decay factor, leading to the decrease of the fit of NS-LS with the fixed decay factor. Thus, since non-traditional monetary policy seems to have an effect on the decay factor, NS-LS is not good in the small decay-factor environment.

III-3. Effects of Monetary Policy by the Bank of Japan

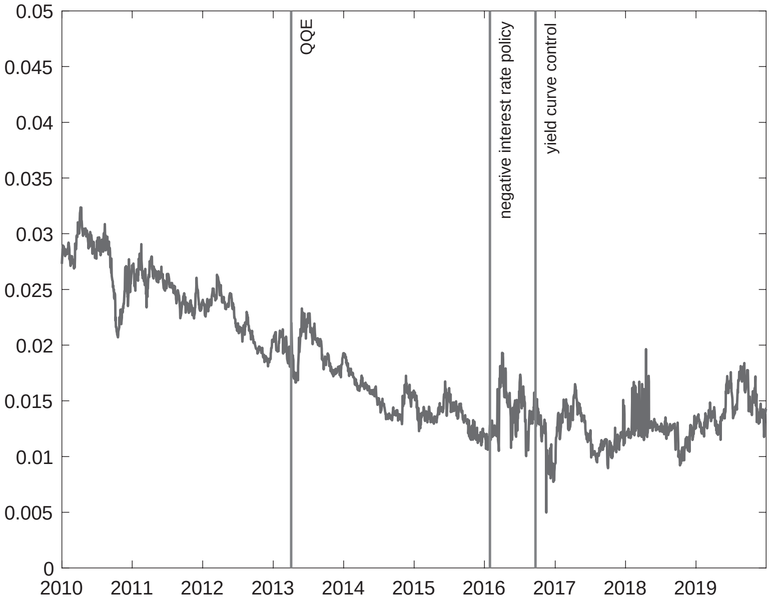
The related literature shows that the decay factor of the NS model is interpreted as one of the monetary policy variables or affected by monetary policy. For example, Okina and Shiratsuka (2004) interpret that the decay factor is the degree of forward guidance. Koeda and Sekine (2021) show that the decay factor depends on the size of the JGB holding of the BOJ and maturities of the yields.

In this study, I investigate how the four yield curve factors obtained from NS-NL are affected by monetary policy of the BOJ. The recent monetary policy of the BOJ is QQE, negative interest rate policy, and yield curve control. The dates when the BOJ introduced them are April 4, 2013, January 29, 2016, and September 21, 2016, respectively.

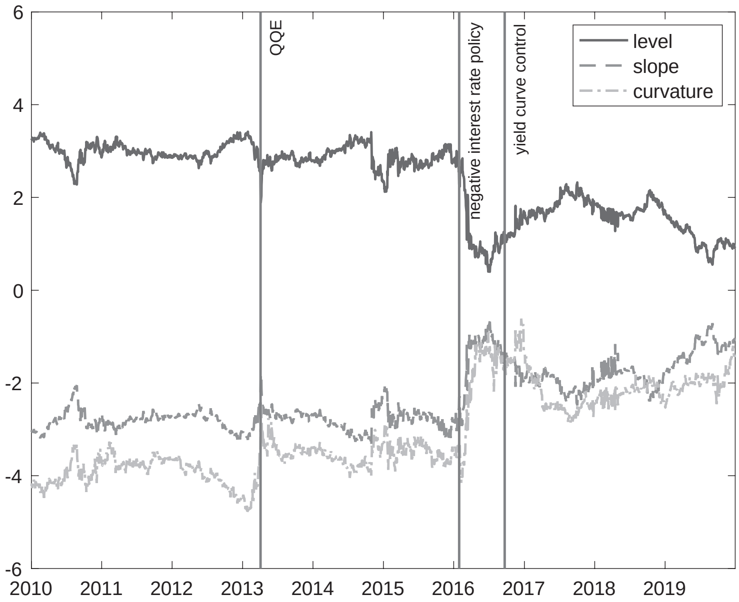
Figure 6 shows how estimates of level, slope, curvature and the decay factor change after the introduction of each policy of the BOJ. Panel a) of the figure shows estimates of the

Figure 6. Effect of monetary policy by the BOJ

a) Decay factor



b) Level, slope, and curvature



Note: This figure shows how the estimates of the four yield curve factors from NS-NL are affected by monetary policy of the BOJ (QQE, negative interest rate policy, and yield curve control). Panel a) indicates the effect on the decay factor, whereas panel b) corresponds to the effect on the level, slope, and curvature.

decay factor obtained from NS-NL. The decay factor slightly increased after the introduction of QQE to July 1, 2013, but afterwards decreased to around 0.015 in the latter of the 2010s. Koeda and Sekine (2021) show that the declined decay factor arises from the increased JGB holding of the BOJ. Also, the volatility of the decay factor decreased after the introduction of yield curve control. Panel b) of the figure shows estimates of level, slope, and curvature obtained from NS-NL. In the panel, the level decreased substantially immediately after the introduction of the negative interest rate policy. The estimated level was 2.281 on the day and below 1 the day after 45 business days. The estimated slope and curvature increased immediately after the introduction of the policy.

IV. Conclusion

In this study, I estimated the yield curves employing the NS model with daily JGB yield data. I compared fits from NS-LS with a fixed decay factor with those from NS-NL under which I estimate the decay factor every period. As a result, NS-NL is better among all of 8,021 periods from April 3, 1989, to December 31, 2019. The F-test indicates that estimating the decay factor every period is recommended especially in the low-interest-rate environment like the one that was observed in the 2010s. Focusing on monetary policy of the BOJ, QQE decreased the estimated decay factor slowly. The introduction of the negative interest policy led to an immediate decrease in level. Also, the volatility of the yield curve factors decreased after the introduction of yield curve control.

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Appendix

In the Appendix, I estimate the Svensson (1994) model, which is the NS model with an additional curvature and decay factor. The Svensson model is

$$y(\tau) = \beta_0 + \beta_1\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau}\right) + \beta_2\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau}\right) + \beta_3\left(\frac{1-e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau}\right),$$

where β_3 is the second curvature. Since there are two decay factors in the Svensson model, I call the λ_1 “first decay factor,” and λ_2 “the second decay factor.”

The estimation process is the same as the NS model. First, I estimate the Svensson model with monthly yield data employing the state-space model. In this estimation, I obtain two estimated decay factors. Next, I estimate the model by OLS with the two decay factors using daily yield data. Lastly, I estimate the model by nonlinear OLS using the estimated yield curve factors and two decay factors as initial values. Here, I call the Svensson model with OLS “SV-LS,” and the Svensson model with nonlinear OLS under which I estimate the two decay factors every period “SV-NL.”

Figure A1 shows the estimates of the two decay factors obtained from SV-NL. The two decay factors from the state-space model are 0.0390 and 0.0228, respectively. This figure shows estimates of the two decay factors from SV-NL under which I use the estimates of the yield curve factors and these decay factors obtained from SV-LS as initial values. Compared with estimates of decay factor from NS-NL shown in Figure 2, those from SV-NL fluctuate largely. Also, the values of the two decay factors are relatively close to each other since the identification would be difficult.

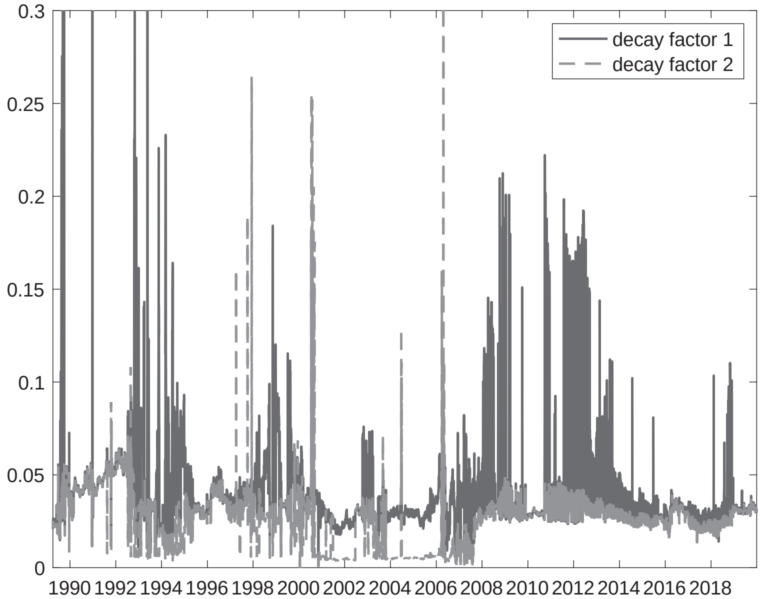
Figure A2 shows estimates of level, slope, and curvature obtained from SV-LS and SV-NL, respectively. Since I fix the decay factor in estimating SV-LS, volatility of the other yield curve factors is relatively small. The estimates of level and curvature of SV-LS are not so far away from those of the NS models. The estimates of the two curvatures are almost symmetric with respect to the horizontal line of 0. However, the estimates of level, slope, and curvature of SV-NL are very volatile. European Central Bank (ECB) publishes the estimates of the yield curve factors of the Svensson model with the daily data.[3] Like my estimates of the Svensson model, those computed by ECB are also volatile.

Table A1 compares fitted yields computed from SV-LS with those from SV-NL by conducting the F-test. The null hypothesis of the test is λ_1=0.0390 & λ_2=0.0228 because I fix the two decay factors of SV-LS at 0.0390 and 0.0228, respectively. Compared with the results of the NS model shown in Table 1, the rejection rate is lower every decade. This result indi-

[3] As for the estimates obtained from the Svenson model, see the website of ECB.
https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html

cates that nonlinear OLS is not positively recommended in estimating the Svensson model. The reason seems to be related to the difficulty of estimating the Svensson model with nonlinear OLS.

Figure A1. Evolution of decay factor estimated from Svensson model



Note: This figure shows estimates of the two decay factors estimated from the Svensson model.

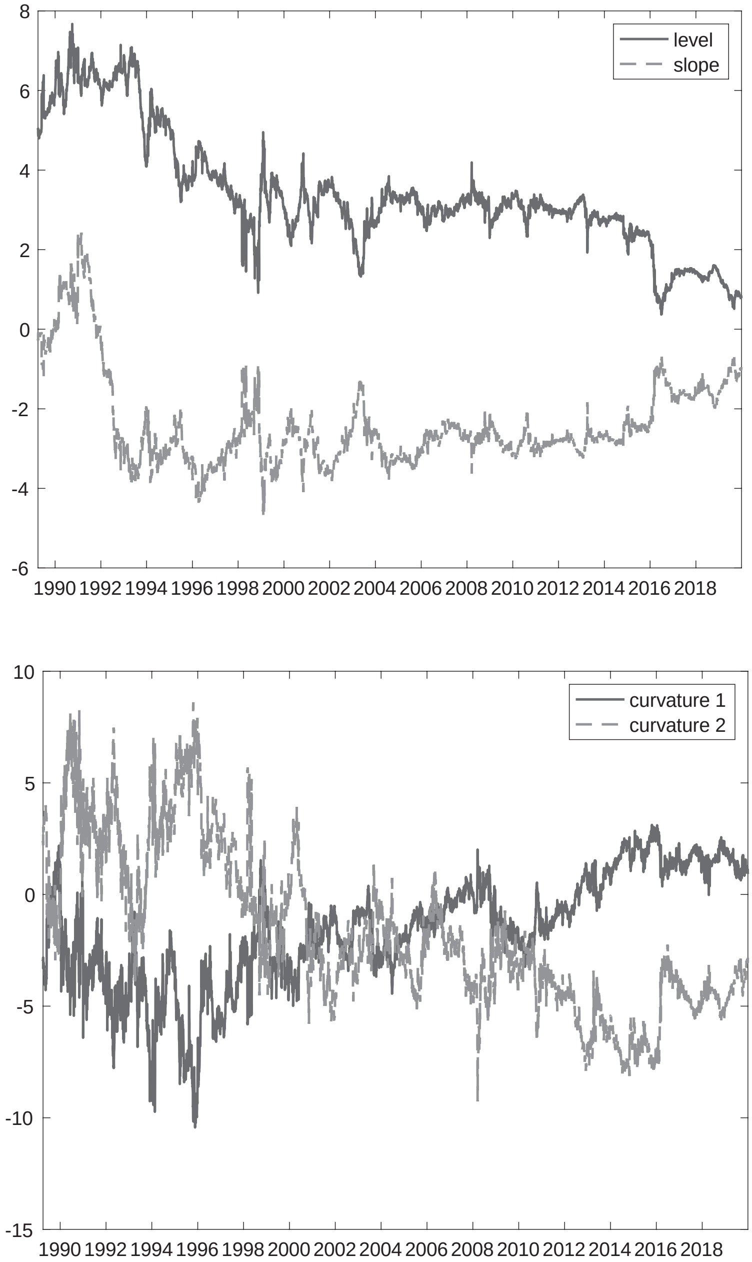
Table A1. Evaluation of the two Svensson models by F-test

	1989-1999	2000-2009	2010-2019	1989-2019
1% significance level	0.066	0.166	0.039	0.090
5% significance level	0.080	0.276	0.186	0.178
10% significance level	0.087	0.342	0.247	0.222

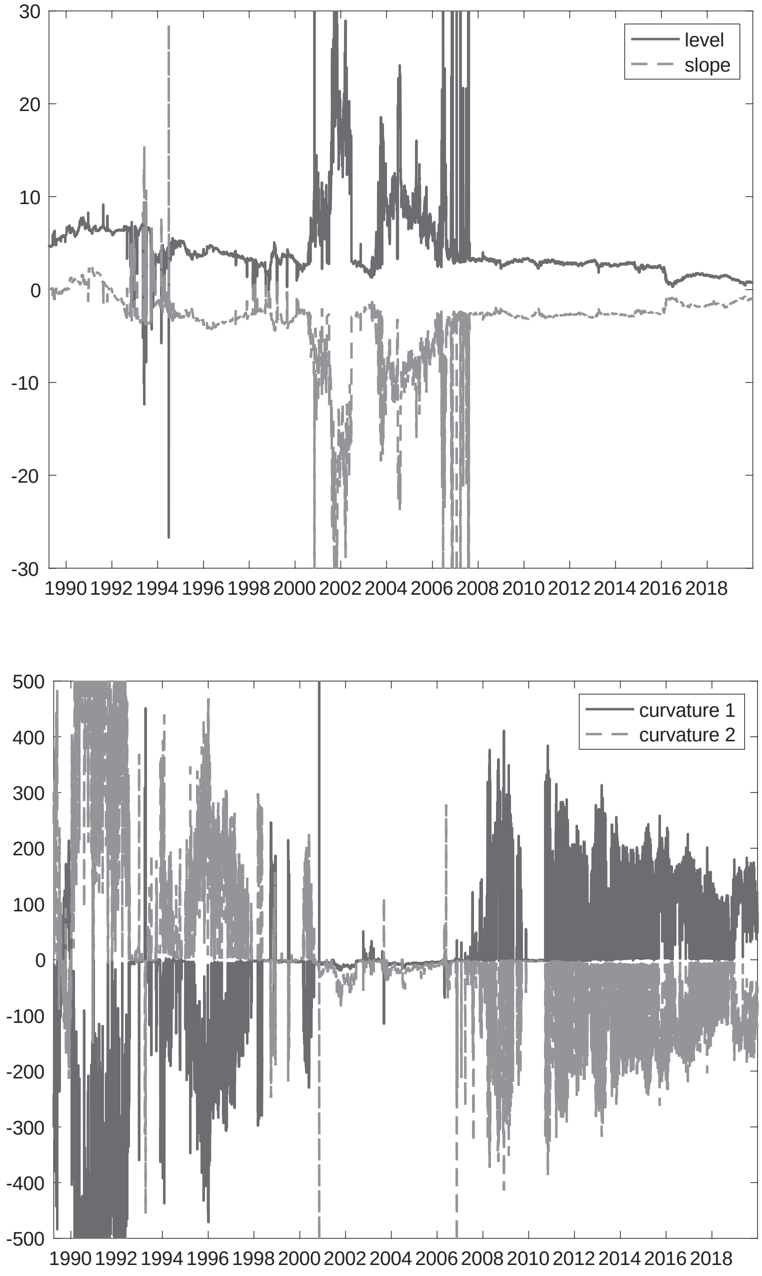
Note: This table shows the results of the rejection ratio by the F-test to compare the fit of SV-LS and SV-NL. The null hypothesis is λ_1=0.0390 & λ_2=0.0228 because the two decay factors are fixed at 0.0390 and 0.0228 in SV-LS, respectively. Here, 1989-1999, 2000-2009, and 2010-2019 indicate April 3, 1989, to December 31, 1999, January 3, 2000, to December 31, 2009, and January 1, 2010, to December 31, 2019, respectively.

Figure A2. Evolution of level, slope, and curvature estimated from Svensson model

a) SV-LS



b) SV-NL



Note. This figure shows estimates of the level, slope, and two curvatures obtained from the Svensson model. Panel a) indicates the estimates from SV-LS, whereas panel b) corresponds to those from SV-NL.