Incomplete Market and Optimal Debt in an Economy with an Overlapping Generation Structure*

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Abstract

Aiyagari and McGrattan (1998) point out that in an incomplete market model, government debt may help to level out consumption by serving as a means of households savings and thereby improve social welfare. Aiyagari and McGrattan state that issuing a certain quantity of government debt is an optimal option. A similar analysis is conducted by Nakajima and Takahashi (2017) with respect to Japan. Aiyagari and McGrattan's model assumes an infinite period of life for individuals. However, Peterman and Sager (2018) point out that if a finite period of life is assumed for individuals, the optimal quantity of government debt is a negative figure because demand for savings declines compared with the case of individuals with an infinite period of life. In addition, in Aiyagari and McGrattan's argument, the comparison is conducted only under a stationary equilibrium and the cost that arises during the transition path in which the debt quantity changes is overlooked. This paper conducts a review of existing literature on those two points.

Keywords: government debt, incomplete market, overlapping generation structure, transition path

JEL Classification: H6, E21, E6

I. Introduction

This paper reviews an analysis of the optimal amount of government debt using an incomplete market model in which individuals are exposed to shocks for which there is no insurance market, resulting in excess savings. In Takahashi (2021), a similar review is conducted in this journal, where the stationary equilibrium analysis in an incomplete market model with individuals living for an infinite period was mainly explained. In this paper, we focus on (1) a model with individuals living for a finite period and (2) the analysis of transi-

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tion paths.

Aiyagari and McGrattan (1998) (hereinafter called AM) point out that when individuals face shocks that cannot be covered by the insurance market, government debt may improve social welfare by providing a means for saving because of the existence of excess savings. Later, Floden (2001) analyzes the case where not only government debt but also income transfer is added as a policy instrument. Nakajima and Takahashi (2017) apply the AM analysis to Japan and point out that the government may not be able to position debt holding as an optimal policy in Japan, where the risk faced by each individual is small, unlike the case in the United States.

Peterman and Sager (2018) (hereinafter called PS) introduce an overlapping generation structure into an incomplete market model and obtain results that differ from AM's conclusion that the optimal policy for the government is to save rather than incur debt.

All of the literature mentioned so far compares a stationary equilibrium under different levels of government debt. Suppose that the government debt ratio is reduced from over 200% to 60% of GDP. Then the tax rate will be higher in the former case because of the need to pay interest on the government debt if comparison is made only at stationary equilibrium, but the tax rate will be higher in the short run in order to reduce the government debt if the transition from the former to the latter is considered. Therefore, simply comparing a stationary equilibrium with both high government debt ratio and high tax rate to a stationary equilibrium with both low government debt ratio and low tax rate misses the cost of short-term fiscal consolidation. Röhrs and Winter (2017) address this issue by analyzing the transition path in an incomplete market model. Ino and Kobayashi (2020) include the transition path in their analysis of whether to raise the consumption tax early or postpone it in order to reduce government debt.

In order to understand the position of existing studies, we can classify them into four quadrants based on the two axes of "models of individuals who live for an infinite period/ models of individuals who live for a finite period" and "analyses of stationary equilibrium only/analyses that takes into account the transition path" as shown in Tables 1 and 2.

Table 1. Analyses of government debt using the incomplete market model: United States

USA	Infinitely lived individuals	Finitely lived individuals
Stationary equilibrium	Aiyagari and McGrattan (1998), Floden (2001)	Peterman and Sager (2018)
Transition path	Röhrs and Winter (2017)	

Table 2. Analyses of government debt using the incomplete market model: Japan

Japan	Infinitely lived individuals	Finitely lived individuals
Stationary equilibrium	Nakajima and Takahashi (2017)	
Transition path	Ino and Kobayashi (2020)	

This paper first discusses the basic incomplete market model introduced by AM. Then in Chapter 3, we introduce PS, which introduce an overlapping generation structure; in Chapter 4 and 5, we review Röhrs and Winter (2017) and Ino and Kobayashi (2020) as analyses dealing with transition paths and conclusions are drawn in Chapter 6.

II. Government Debt in an Incomplete Market Model: A Stationary Equilibrium Comparison with Individuals Living for an Infinite Period

AM point out that in an incomplete market model, where individual labor productivity is subject to idiosyncratic shocks but there is no insurance against them, the existence of government debt may improve social welfare by serving a function to absorb excess savings generated by precautionary motives. The details of this model are explained below.

II-1. The model: Aiyagari and McGrattan (1998)

There are three types of sectors in this model: households, firms, and the government. Firms hire labor L_t, and borrow K_t from households to produce goods. The production function is formulated as $Y_t = F(K_t, z_t L_t)$ where z_t is the labor-augmenting technological progress. To match the economic growth in the data, it is assumed to grow at a constant rate, as in $z_t = z(1+g)^t$ where g is the rate of technological progress.

Households own one unit of time each period, use l_t for leisure, and spend the rest 1-l_t on labor to earn wage incomes. Households face the idiosyncratic labor productivity shocks e_t, so given the wage rate w_t, their labor income is given by $w_t e_t(1-l_t)$. They can also rent their asset to either firms or government and earn interest rate income r_t. In addition, they receive income transfer Tr_t. Households will allocate these incomes to consumptions c_t and savings a_{t+1}. So the budget constraint of households is given by

$$c_t + a_{t+1} \leq (1-\tau_y) w_t e_t (1-l_t) + (1 + r_t(1-\tau_y)) a_t + Tr_t$$

where τ_y is the tax rate on income.

Households' utility function is given by $u(c_t, l_t) = \dfrac{(c_t^{\mu}, l_t^{1-\mu})^{1-v}}{1-v}$. Given the initial condition (a_0, e_0), households choose $\{c_t, l_t, a_{t+1}\}_{t=0}^{\infty}$ to solve

$$\max_{\{c_t, l_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{(c_t^{\mu}, l_t^{1-\mu})^{1-v}}{1-v} \tag{1}$$

$$\text{s.t. } c_t + a_{t+1} \leq (1-\tau_y) w_t e_t (1-l_t) + (1 + r_t(1-\tau_y)) a_t + Tr_t \tag{2}$$

$$c_t \geq 0, a_{t+1} \geq 0, \forall t.$$

The main feature of this model is that the market is incomplete because there is no insurance market for the labor productivity shock e_t. Households that aim to equalize consumption in order to maximize utility will respond to this situation by increasing their savings.

The government receives income taxes from households and issues government debt. Then, the government uses them for the government purchase G_t, repayment of the debt, and income transfer. So the government budget constraint is given by

$$G_t + (1+r_t) B_t + Tr_t = B_{t+1} + \tau_y r_t K_t + \tau_y w_t L_t.$$

We describe how to remove the trend and the definition of stationary equilibrium in the appendix.

AM set the parameter values as in Table 3.

Table 3. Parameter values of the incomplete market model in AM

Description of parameters	Parameter	Value	Target
Growth rate	g	1.85%	Average of United States after WW2
Govt. expenditure	γ	21.7%	Average of United States after WW2
Govt. Transfer	χ	8.2%	Average of United States after WW2
Govt. Debt	b	2/3	Assumption
Discount factor	β	0.991	Interest rate
Borrowing limit	$\underline{a}$	0.0	Assumption
Capital share of income	θ	0.3	NIPA
Depreciation rate	δ	0.075	Investment/Output=0.255
Coef. on relative risk aversion	μ	1.5	Literature
Labor disutility	η	0.328	Labor elasticity
AR coef. of shock	ρ	0.6	Literature
Standard deviation of shock	σ	0.3	Literature

AM compare stationary equilibria at various values of the government debt-to-GDP ratio. Figure 1 shows the simulation results.

When government debt is zero, interest rates are lowered due to excess savings, and the economy as a whole is in a situation of excess capital because individuals do not stop saving for precautionary savings even in such a situation. As a result, the number of borrowers of savings increases as government debt increases, and social welfare may be improved by eliminating excess savings. After setting parameters to adapt their model to the data of the U.S. economy, they conclude that the government debt-to-GDP ratio that maximizes social welfare is about 66%, which is not so different from the level of the U.S. at that time, as shown in Figure 1.

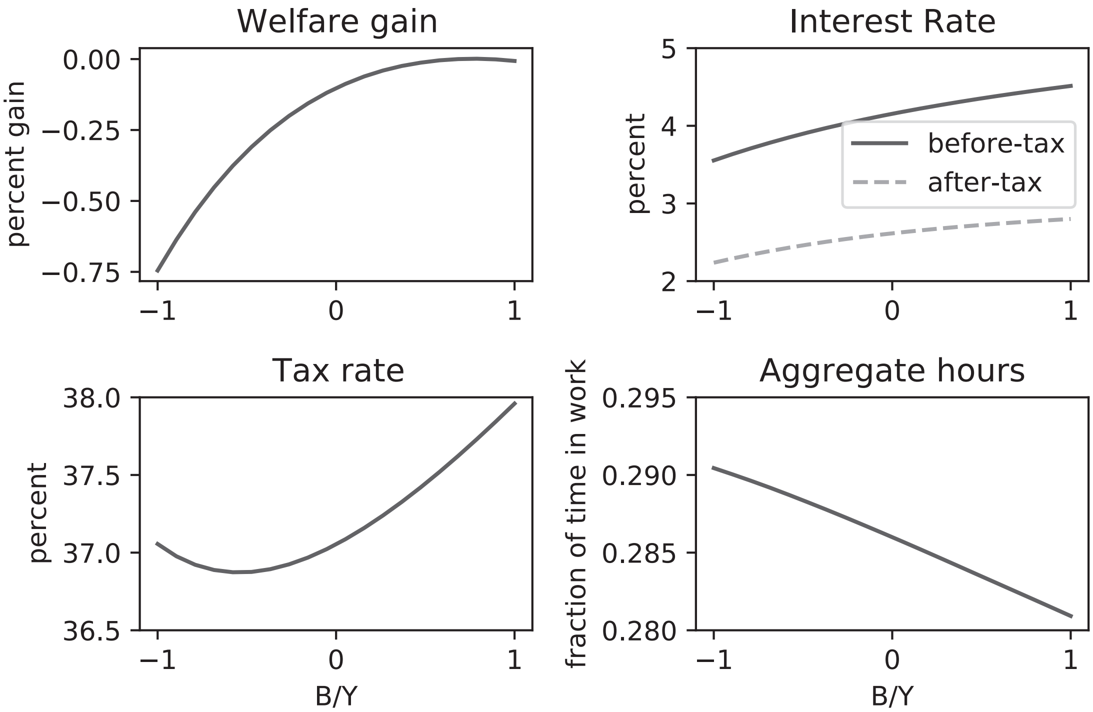
III. Government Debt in an Incomplete Market Model: A Stationary Equilibrium Comparison with Individuals Living for a Finite Period

In contrast to AM's conclusion that a positive government debt-to-GDP ratio maximizes social welfare, PS use an incomplete market model that introduces an overlapping generation structure in which individuals live for only a finite period to conclude that the government debt-to-GDP ratio that maximizes social welfare should be negative, i.e., the government should have a net worth.

III-1. Peterman and Sager: Households

The biggest difference between AM and PS is that households live for a finite period in

Figure 1. Stationary equilibrium comparison in AM



PS. In this economy, there are J generations, with $j=1$ corresponding to 21 years old and $j=80$ corresponding to 100 years old. The probability that households with age j survives to $j+1$ is denoted as ψ_j, and set $\psi_J=0$. In every period, a new generation j=1 is born, and the population of that generation grows at a constant rate $g_n>0$.

The preference of households with age $j=0$ is given by

$$\mathbb{E}_1 \sum_{j=1}^{J} \beta^{j-1} \psi_j [u(c_j) - \upsilon(h_j)].$$

Individual labor productivity e_j evolves according to

$$\log(e_j) = \kappa + \theta_j + v_j + \varepsilon_j$$

where (i) $\kappa \sim N(0, \sigma_\kappa^2)$ is the individual-level fixed effects, (ii) $\{\theta_j\}_{j=1}^{J}$ is non-stochastic age level fixed effects, (iii) v_j is an individual-level persistent shock which follows AR(1)$v_{j+1} = \rho v_j + \eta_{j+1}$, $\eta_{j+1} \sim N(0, \sigma_v^2)$, and (iv) $\epsilon_j \sim N(0, \sigma_\epsilon^2)$ is an individual-level temporally shock. To simplify the notation, we collect these shocks and write $\varepsilon_j = (\kappa, \theta_j, v_j, \epsilon_j)$. Let $\pi_j(\varepsilon_{j+1}|\varepsilon_j)$ denote the transition probability.

III-2. Peterman and Sager: Social Security

PS introduce the social security to have an overlapping generation structure close to the data. When households receive labor income *weh*, either s/he should pay amount multiplied by tax rate τ_{ss} or upper limit $\overline{m}$ as a social security premium. PS assume that half of the social security cost is paid by households and the rest is paid by firms. So, the social security cost households pay is given by $(\tau_{ss}/2)\min\{weh, \overline{m}\}$.

Pension benefits after retirement are determined by households' average labor income. Given $m_1 = 0$, average labor income evolves according to

$$m_{j+1} = \begin{cases} \dfrac{1}{j}(\min\{weh, \overline{m}\} + (j-1)m_j) & \text{for } j \leq 35, \\ \max\{m_h, \dfrac{1}{j}(\min\{weh, \overline{m}\} + (j-1)m_j)\} & \text{for } j \in (35, J_{ret}), \\ m_j & \text{for } j \geq J_{ret}. \end{cases} \tag{21}$$

The amount of pension benefits is determined by paying a fixed percentage of the average salary, which varies with income. If the average salary is less than b_1^{ss}, the pension benefits are paid at a rate of τ_{r1}; if the average salary exceeds b_1^{ss}, the pension benefits are paid at a rate of τ_{r2}; if the average salary exceeds b_2^{ss}, the pension benefits are paid at a rate of τ_{r3}; and thereafter, the pension benefits do not increase even if the average salary increases. This relationship can be written down as

$$b_{base}^{ss}(mJ_{ret}) = \begin{cases} \tau_{r1} mJ_{ret} & \text{for } mJ_{ret} \in [0, b_1^{ss}), \\ \tau_{r1} b_1^{ss} + \tau_{r2}(mJ_{ret} - b_1^{ss}) & \text{for } mJ_{ret} \in [b_1^{ss}, b_2^{ss}), \\ \tau_{r1} b_1^{ss} + \tau_{r2} b_2^{ss} + \tau_{r3}(mJ_{ret} - b_1^{ss} - b_2^{ss}) & \text{for } mJ_{ret} \in [b_2^{ss}, b_3^{ss}), \\ \tau_{r1} b_1^{ss} + \tau_{r2} b_2^{ss} + \tau_{r3} b_3^{ss} & \text{for } mJ_{ret} \geq b_3^{ss}. \end{cases} \tag{22}$$

Under this social security system, taxable income is defined as

$$y(h, a, \varepsilon) = \begin{cases} we(\varepsilon)h + r(a + Tr) - \dfrac{\tau_{ss}}{2}\min\{we(\varepsilon)h, \overline{m}\} & \text{if } j < J_{ret} \\ r(a + Tr) & \text{if } j \geq J_{ret}. \end{cases} \tag{23}$$

Given the taxable income y, we assume that income tax is given by the functional form following Gouveia and Strauss (1994):

$$Y(y) = \tau_0(y - (y^{-\tau_1} + \tau_2))^{-\frac{1}{\tau_1}}. \tag{24}$$

III-3. Peterman and Sager: Optimization Problem of Households

The state variables of households consist of asset a, labor productivity ε, social insurance premiums m and age j. The optimization problem of households with age j before retirement is

$$V_j(a, \varepsilon, m) = \max_{c, a', h}[u(c) - v(h)] + \beta\psi_j \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon)V_{j+1}(a', \varepsilon', m') \tag{25}$$

$$\text{s.t. } c + a' \leq we(\varepsilon)h + (1+r)(a + Tr) - \frac{\tau_{ss}}{2}\min\{we(\varepsilon)h, \overline{m}\} - Y(y(h, a, \varepsilon)) \tag{26}$$

$$a' \geq \underline{a}. \tag{27}$$

After retirement,

$$V_j(a, \varepsilon, m) = \max_{c,\, a'}[u(c) - v(0)] + \beta\psi_j \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon) V_{j+1}(a', \varepsilon', m') \tag{28}$$

$$\text{s.t. } c + a' \leq (1+r)(a+Tr) + b_{ss}(m) - \Upsilon(r(a+Tr)) \tag{29}$$

$$a' \geq \underline{a}. \tag{30}$$

Let λ (a, ε, m) denote the probability density function of households over (a, ε, m), and let μ_j denote the fraction of households with age j. The definition of stationary equilibrium is in the appendix. Table 4 shows the parameter settings of this model.

Table 4. The parameter values in PS

Description of parameters	Parameter	Value	Target
Maximum age	J	81	Assumption
Population growth	g_n	1.1%	Conesa, Kitao, and Krueger (2009)
Coef. on relative risk aversion	σ	2.0	Kaplan (2012)
Frisch elasticity	γ	0.5	Kaplan (2012)
Coef. on dis-utility on labor	χ_1	56.2	Average hours worked = 1/3
Discount factor	β	1.012	Capital/Output=2.7
Borrowing limit	$\underline{a}$	0.0	Assumption
Capital share of income	α	0.36	NIPA
Depreciation rate	δ	0.0833	Investment/Output=0.255
Growth rate	g_y	1.85%	NIPA
AR coef. of persistent shock	ρ	0.958	Kaplan (2012)
Variance, persistent shock	σ_ν^2	0.017	Kaplan (2012)
Variance, permanent shock	σ_κ^2	0.065	Kaplan (2012)
Variance, temporally shock	σ_ϵ^2	0.958	Kaplan (2012)
Govt. expenditure	G/Y	0.155	NIPA 1998-2007Average
Govt. debt	B/Y	0.667	NIPA 1998-2007 Average
Coef. on income tax	τ_0	0.258	Gouveia and Strauss (1994)
Income tax	τ_1	0.786	Gouveia and Strauss (1994)
Income tax	τ_2	4.648	Govt budget balance
Social security tax	τ_{ss}	0.103	Social security

III-4. Numerical Analysis

Under these parameter settings, the optimal government debt-to-GDP ratio in the PS is −61%, which means that the government should not hold net debt but should save, unlike the case of individuals who live for an infinite period.

In AM's model, since each individual lives for an infinite period, accumulating savings can maintain consumption smoothing against shocks for a long period of time. Therefore, raising the interest rate by increasing the government debt and encouraging individuals to save has a positive effect on social welfare. On the other hand, in PS's model, the benefits of increasing government debt and interest rates are not significant because individuals live for a finite period, and most of that time is spent in their youth when accumulate savings, and in old age when they withdraw their savings. Rather, the results suggest that government savings can improve social welfare by lowering interest rates and making consumption more equal by lowering private savings.

IV. Government Debt in an Incomplete Market Model: A Study of Transition Paths

Röhrs and Winters (2017) (hereinafter called RW) analyze the transition path of government debt reduction in an incomplete market model under the assumption of individuals living for an infinite period. The AM analysis compares the stationary equilibrium under the different government debt-to-GDP ratio and determined the optimal government debt that maximizes the social welfare. However, this analysis does not immediately lead to the conclusion that the current government debt-to-GDP ratio should be shifted to the optimal level, no matter what the current ratio is. It is because if the current government debt ratio was 200%, a large amount of fiscal consolidation would be required to reach the stationary equilibrium of 66%. The RW analysis takes this into account when discussing the cost of reducing government debt.

RW calculates the transition path when the government debt-to-GDP ratio decreases from 0.66 to 0.6. RW consider two types of decrease in government debt: a convex decrease with a large initial decrease, and then a gradual decrease, and a concave decrease with a small initial decrease and a larger decrease as time passes. In order to reduce the government debt, the tax rate on income τ_y, is varied. Since the welfare loss is smaller in the concave case, this suggests that it is desirable to take a long time span to reduce the government debt in order to give households time to adjust. In addition, even though the welfare at the end of the period is high when only the stationary equilibrium is compared, all the changes in welfare are negative when the transition path is included in the analysis, indicating that it is important to include the analysis of the transition path.

V. Timing of Government Debt Reduction through a Consumption Tax Hike

Following the two postponements of the consumption tax hike from 8% to 10% in Japan, Ino and Kobayashi (2020) analyzed how the choice of whether to raise the consumption tax early or after the increase of government debt differs between individuals with different asset holdings. The model is for individuals living for an infinite period, similar to AM, but the transition path is calculated as follows.

V-1. Transition Path

Let T denote the period when the economy reaches the new stationary equilibrium after the policy change. After the period T, the economy is in the stationary equilibrium forever, so the transition path can be characterized as a sequence of functions with length T. We assume that the economy reaches the stationary equilibrium associated with fiscal policy τ^{ini} initially, and compute the transition path to the new stationary equilibrium associated with fiscal policy $\tau^{terminal}$. We define the transition path in the appendix.

V-1-1. Numerical Analysis

The parameters of this model are set to be consistent with the Japanese data. The specific settings are as follows.

In this model, one period in the model corresponds to one year.

Following Nakajima and Takahashi (2017), we choose the value of discount factor and exogenous growth rate to match the recent low growth and low interest rate: $\beta = 0.991$, $g = 0.009$. We use the standard utility function $u(c, l) = \dfrac{(c^{\eta} l^{1-\eta})^{1-\mu}}{1-\mu}$ and set $\mu = 1.5$, $\eta = 0.328$. We assume that households cannot borrow, that is, $\underline{a} = 0$. To set the capital's share of income to 0.3, we set $\alpha = 0.3$. We set $\delta = 0.075$ so that the output-to-capital ratio is close to 4.

We assume that the labor productivity shock follows the AR(1):

$$\log(\epsilon_{t+1}) = \rho \log(\epsilon_t) + e_t, \quad e_t \sim N(0, \sigma_e^2).$$

In Nakajima and Takahashi (2017), a fixed-effects part is included for each individual throughout time, but in this paper, the fixed-effects part is removed to simplify the calculation because we are calculating the transition path. For the parameter setting, we follow Nakajima and Takahashi (2017) and set ρ=0.9, σ=0.226. The shocks themselves are continuous variables, but in order to implement the calculation, we use the method of Tauchen (1986) to approximate them by a discrete Markov process with seven possible values.

Regarding parameters associated with fiscal policy, for the initial stationary equilibrium, we set $\chi = \dfrac{Tr}{Y} = 0.14$, $\gamma = \dfrac{G}{Y} = 0.13$, and for the terminal stationary equilibrium, we set $\gamma = \dfrac{G}{Y} = 0.24$. For the initial stationary equilibrium, we choose this value so that the consumption tax rate is equal to 8%, and for the terminal stationary equilibrium, we follow Fukawa and Sato (2009). For tax rates set exogenously, we follow Hansen and İmrohoroğlu (2016) and set $\tau_y = 0.34$.

V-2. Results: Transition Path

Ino and Kobayashi (2020) analyze the transition path for policy changes in response to a permanent increase in government spending that is not expected to occur in period 0. Specifically, we run the following simulation.

In the initial period, the economy is assumed to be in a stationary equilibrium with $b = 0.6$, i.e., the government debt-to-GDP ratio is 60%.

In period 0, a permanent increase in government spending g occurs. Because of the increase in spending, if τ_y is fixed, the consumption tax τ_c must be increased or the government bond issuance b_{t+1} must be increased to satisfy the government's budget constraint equation.

Under these circumstances, we consider the following two policy scenarios.

1. The consumption tax rate is kept at 8% until the government debt-to-GDP ratio reach-

es 150% ($b=1.5$), and then the consumption tax τ_c is set so that the government debt decreases to $b=1$ over 40 years.

2. The consumption tax rate is kept at 8% until the government debt-to-GDP ratio reaches 200% ($b=2$), and then the consumption tax τ_c is set so that the government debt decreases to $b=1$ over 40 years.

The first scenario corresponds to an early tax hike, while the second corresponds to the postponement of the tax hike. In both scenarios, the consumption tax rate after the government debt ratio reaches a certain level is set so that b converges to $b=1$ at a certain rate over 40 years.

By calculating the transition path under these two policy scenarios and calculating which scenario households would have higher utility in period 0, we examine the preferences of individual households for an early tax hike.

Figure 2 plots the transition path under these policy scenarios.

As long as the budget deficit is covered by government debt, interest rates will rise and capital will decline. This is because more debt raises the demand for assets. Once the consumption tax begins to increase, government debt decreases and capital increases.

Since the government debt-to-GDP ratio at the endpoint is greater than at the beginning, interest rates will decrease in the long run.

Comparing the early and postponed tax hike scenarios, we can see that the postponed scenario would require a larger consumption tax hike. This is because postponing the tax hike will increase the outstanding balance of government debt, and because higher interest rates will require greater financial resources to reduce government debt. However, postponing the tax hike means that the consumption tax hike will occur further in the future, so the fact that the amount of the consumption tax hike is large does not necessarily mean that an early tax hike is immediately desirable.

Figure 3 plots the difference in the value function of each individual under an early tax hike and postponement of the tax hike. For each labor productivity level ϵ, this figure plots

$$V_0(a,\epsilon;\ \textit{early tax increase}) - V_0(a,\ \epsilon;\ \textit{late tax increase}). \tag{50}$$

If this figure is positive, households in the corresponding state (a, ϵ) will gain higher utility in the early tax hike scenario.

This figure shows that households with fewer assets and lower labor productivity will gain higher utility from an early tax hike, while households with more assets and higher labor productivity will gain higher utility from postponing the tax hike.

This is because households with more assets can earn more interest income as government debt increases and interest rates rise due to the postponement of the tax hike, while households with fewer assets cannot enjoy the benefits of the tax hike and face the disadvantages of the higher consumption tax due to the postponement of the tax hike. In addition, households with high labor productivity but low assets can save and immediately increase their asset holdings, so they will gain a higher utility from a rise in interest rates due to the postponement of the tax hike, whereas households with low labor productivity will withdraw their savings and gain a higher utility from an earlier tax hike even if they currently

Figure 2. Transition path: early tax hike *b*=1.5 (solid line) late tax hike *b*=2 (dotted line)

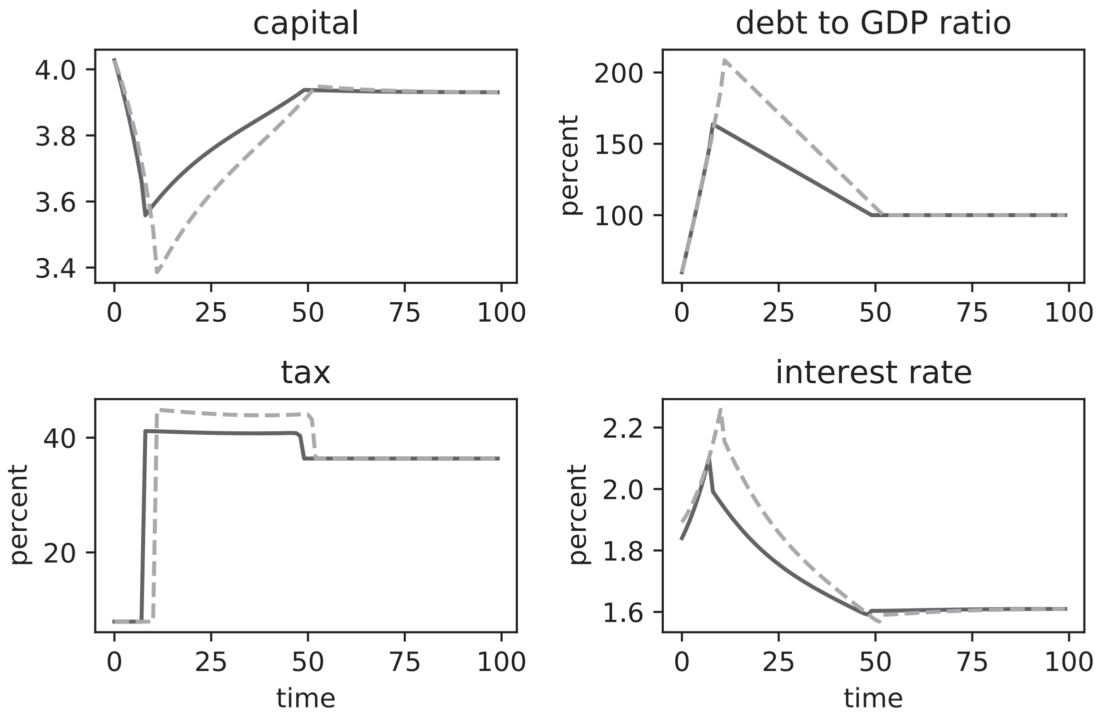
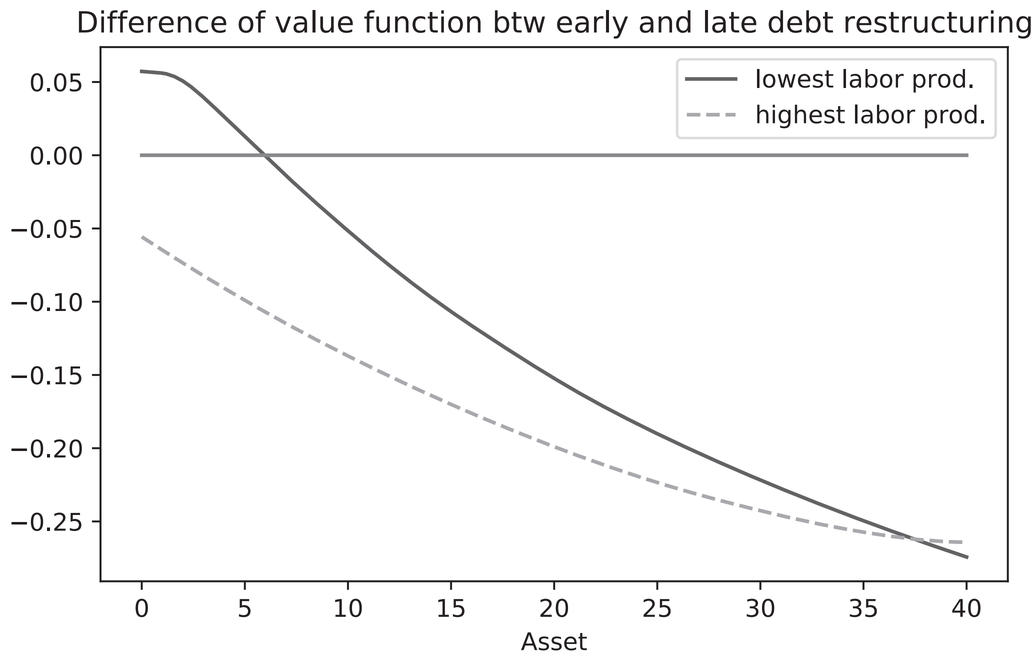


Figure 3. The difference of utility between early and late tax hike

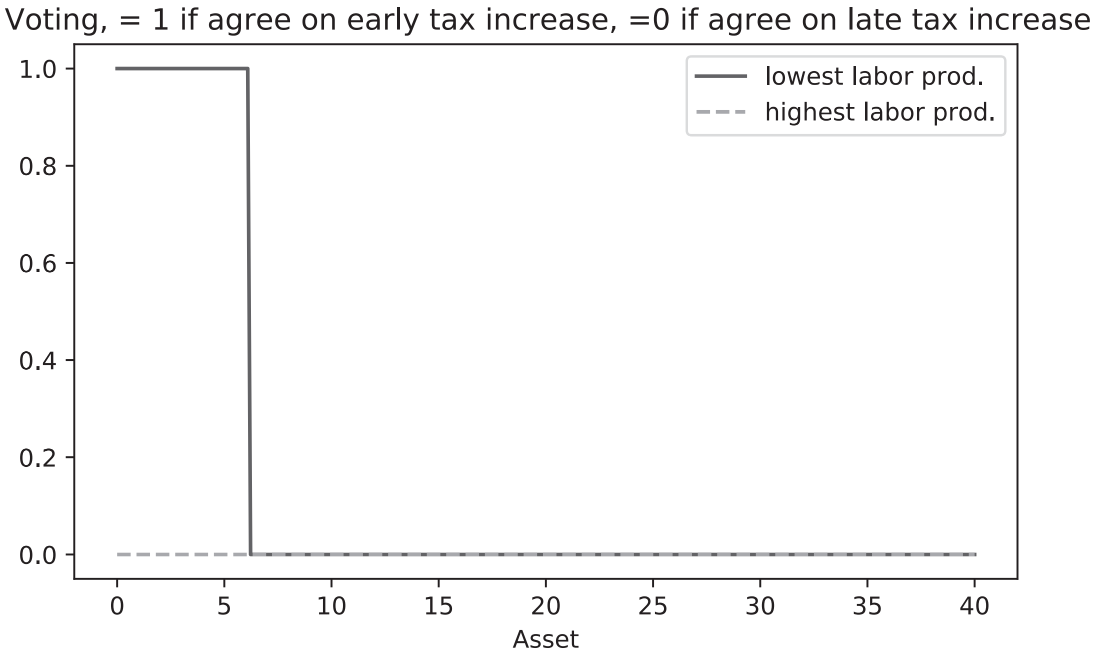


hold a lot of assets. Figure 4 shows the effect of tax hikes on labor productivity.

In Figure 4, we plot the voting behavior of households in each labor productivity to the policy scenario. This value is 1 if the difference in the value function shown in equation (50) is positive, and 0 otherwise, and can be seen as an indicator of whether or not the households will vote for an early tax hike.

Figure 4 shows that households with fewer assets and lower labor productivity vote for an early tax hike, while households with more assets and higher labor productivity vote for the postponement of the tax hike, according to the difference in the value function. Once the voting behavior of individual households is known, it can be multiplied by the distribution of households and added up to calculate the overall voting rate for an early tax hike, which is 60.01% under the Ino and Kobayashi (2020) calibration. The high voter turnout for an early tax hike is due to the fact that most households are located near the lower end of the distribution of asset holdings.

Figure 4. Voting: =1 if agree on early tax hike, =0 if agree on postponing tax hike.



VI. Conclusion

This paper reviews an incomplete market model in which excess savings and positive government debt may increase social welfare, particularly the cases when an overlapping generation structure is introduced and the transition path to a new stationary equilibrium with reduced government debt is analyzed.

Peterman and Sager (2018), assuming individuals who live for a finite period, conclude that the optimal government debt-to-GDP ratio can be negative, unlike the conclusion of Aiyagari and McGrattan (1998) who assume individuals who live for an infinite period. This is because the benefit of increasing the interest rate by increasing the government debt to create an environment in which individuals can easily save can only be enjoyed by the old generations who have finished saving under the assumption individuals living for a finite period, and it is more effective to equalize consumption and leisure by lowering the interest rate.

By analyzing the transition path, Röhrs and Winters (2017) show that even if the equilibrium corresponding to less government debt achieves higher social welfare in the long-term, when the short-term costs of transitioning from a stationary equilibrium with high government debt to a stationary equilibrium with low government debt are included, reducing government debt may not improve social welfare. It should be noted, however, that this result is applicable to the transition from one stationary equilibrium to another, but does not evaluate the transition from an economy with a divergent path of ever-increasing debt to a stationary equilibrium with certain debt, as is the case in Japan today. Kobayashi and Ueda (2021) used a representative individual model to analyze policies to shift from a divergent path of government debt to a stationary equilibrium with a consumption tax hike, and found that fiscal consolidation with a consumption tax hike improved social welfare more than the divergent path.

Ino and Kobayashi (2020) analyze the transition path of whether to raise the consumption tax early or postpone it, and found that people with fewer assets prefer to raise the tax early, which results in a smaller consumption tax hike, while people with more assets prefer to postpone the tax hike, which results in higher interest rates and more interest income.

Analysis using incomplete market models can take income and asset inequality into account and can explain the persistently low interest rates that are a major feature of the current economic environment. On the other hand, due to its computational cost, it is often subjected to simplifying assumptions, such as analysis only between stationary equilibria or individuals living for an infinite period, which may have a significant impact on the conclusions. As shown in Tables 1 and 2, there are still many analyses to be done in this area. In particular, there is no research in either Japan or the U.S. that analyzes transition paths using models that include an overlapping generation structure, and this is a major issue for future research.

Reference

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Appendix

Removing Trend

In AM's model, there is an exogenous technological growth z_t. We cannot define stationary equilibrium because variables grow at a constant rate. Therefore, we first have to remove the trend in order to define the stationary equilibrium.

In order to remove the trend, we define the variables with tildes as a ratio to GDP:

$$\tilde{c}_t \equiv \frac{c_t}{Y_t},\ \tilde{a}_t \equiv \frac{a_t}{Y_t},\ \tilde{w}_t \equiv \frac{w_t}{Y_t},\ \tilde{K}_t \equiv \frac{K_t}{Y_t},\ \tilde{A}_t \equiv \frac{A_t}{Y_t},\ \tilde{Tr}_t \equiv \frac{Tr_t}{Y_t} \equiv \chi. \tag{3}$$

We can transform the households' budget constraint by dividing by Y_t:

$$\frac{c_t}{Y_t} + \frac{a_{t+1}}{Y_{t+1}}\frac{Y_{t+1}}{Y_t} \leq (1-\tau_y)\frac{w_t}{Y_t}e_t(1-l_t) + (1+r_t(1-\tau_y))\frac{a_t}{Y_t} + \frac{Tr_t}{Y_t} \tag{4}$$

$$\tilde{c}_t + (1+g)\tilde{a}_{t+1} \leq (1-\tau_y)\tilde{w}_t e_t(1-l_t) + (1+r_t(1-\tau_y))\tilde{a}_t + X. \tag{5}$$

By defining $\tilde{\beta} \equiv \beta(1+g)^{\mu(1-v)}$, the utility function of households can be written as

$$\sum_{t=0}^{\infty}\beta^t \frac{(c_t^{\mu} l_t^{1-\mu})^{1-v}}{1-v} = \sum_{t=0}^{\infty}\beta^t Y_t^{\mu(1-v)} \frac{[(c_t/Y_t)^{\mu} l_t^{1-\mu}]^{1-v}}{1-v} \tag{6}$$

$$= Y_0^{\mu(1-v)}\sum_{t=0}^{\infty}\beta^t\left(\frac{Y_t}{Y_0}\right)^{\mu(1-v)}\frac{[\tilde{c}_t^{\mu} l_t^{1-\mu}]^{1-v}}{1-v} \tag{7}$$

$$= Y_0^{\mu(1-v)}\sum_{t=0}^{\infty}\tilde{\beta}^t\frac{[\tilde{c}_t^{\mu} l_t^{1-\mu}]^{1-v}}{1-v}. \tag{8}$$

So, the households' optimization problem can be written as

$$\max_{\{\tilde{c}_t, l_t, \tilde{a}_{t+1}\}_{t=0}^{\infty}} Y_0^{\mu(1-v)}\sum_{t=0}^{\infty}\tilde{\beta}^t\frac{[\tilde{c}_t^{\mu} l_t^{1-\mu}]^{1-v}}{1-v} \tag{9}$$

s.t. $\tilde{c}_t + (1+g)\tilde{a}_{t+1} \leq (1-\tau_y)\tilde{w}_t e_t(1-l_t) + (1+r_t(1-\tau_y))\tilde{a}_t + \chi$ (10)

$\tilde{c}_t \geq 0,\ \tilde{a}_{t+1} \geq 0, \forall t.$ (11)

Stationary equilibrium in AM

Given the fiscal policy $\tau = (\tau_y, \tilde{B})$, a stationary equilibrium in this model is a pair of $(V, \tilde{a}', l, w, r, \mu, \tilde{K}, \tilde{L})$ which satisfies:

1. V is the solution to the following Bellman equation and $(\tilde{a}', l)$ is the associated policy function:

$$V(\tilde{a}, \epsilon, \tau) = \max_{\tilde{c}, \tilde{a}' \geq \underline{a}, l}\left\{\frac{(\tilde{c}^{\eta} l^{1-\eta})^{1-\mu}}{1-\mu} + \tilde{\beta}E[V(\tilde{a}', \epsilon', \tau)]\right\} \tag{12}$$

s.t. $\tilde{c} + (1+g)\tilde{a}' = [1 + (1-\tau_y)r]\tilde{a} + (1-\tau_y)\tilde{w}(1-l)\epsilon + \chi.$ (13)

2. Prices (r, w) and capital and labor input $(K, \tilde{L})$ is consistent with the firm's profit maximization

$$w = z(1-\alpha)\left(\frac{K}{zL}\right)^{\alpha-1} \tag{14}$$

$$\tilde{w} \equiv \frac{w}{z} = (1-\alpha)\left(\frac{\tilde{K}}{L}\right)^{\alpha-1} \tag{15}$$

$$r = \alpha\left(\frac{\tilde{K}}{L}\right)^{\alpha} - \delta. \tag{16}$$

3. Labor and capital markets clear.

$$L = \int \epsilon[1 - l(a,\epsilon)]\, d\mu(a,\epsilon), \tag{17}$$
$$\tilde{A} = \tilde{K} + \tilde{B} \tag{18}$$

4. The distribution of households is constant over time.

$$\mu(a', \epsilon', \tau) = \sum_a \sum_\epsilon 1\{a(a, \epsilon) = a'\} P(\epsilon'|\epsilon)\mu(a, \epsilon, \tau) \tag{19}$$

5. Government budget constraint satisfies

$$\tau_y(r\tilde{K} + w\tilde{L}) = \gamma + \chi + (r - g)\,\tilde{B}. \tag{20}$$

Stationary equilibrium in Peterman and Sager

Given a fiscal policy $\tau = (G,B,B',R)$, a stationary equilibrium in this economy is a pair of functions $(\{V_j, c_j, a'_j, h_j, \lambda_j\}_{j=1}^{J}, w, r, K, L, Tr)$ which satisfies

1. V_j is the solution to the Bellman equation and (c_j, a'_j, h_j) is the associated policy function.
2. Prices (r,w) and capital-labor inputs (K,L) are consistent with the firm's profit maximization

$$w = z(1-\alpha)\left(\frac{K}{zL}\right)^{\alpha-1} \tag{31}$$

$$r = \alpha\left(\frac{\tilde{K}}{L}\right)^{\alpha} - \delta. \tag{32}$$

3. Capital and Labor market clear.

$$L = \sum_{j=1}^{J} \mu_j \int e(\varepsilon) h_j(a, \epsilon, m) d\lambda_j(a, \epsilon, m) \tag{33}$$

$$\sum_{j=1}^{J} \mu_j \int a d\lambda_j(a, \epsilon, m) = K + B \tag{34}$$

4. The distribution of households is constant over time.

$$\lambda_{j+1}(a', \epsilon', m) = \int 1\{a'_j(a,\epsilon,m) = a'\}\psi_j \pi(\varepsilon'|\varepsilon)\lambda_j. \tag{35}$$

5. Government budget constraint satisfies

$$G + (1 + r)B = B' + R \tag{36}$$

where the revenue R is given by

$$R \equiv \sum_{j=1}^{J} \mu_j \int Y(y(h_j(a, \varepsilon, m)), a, \varepsilon) d\lambda_j(a, \varepsilon, m). \tag{37}$$

6. Unanticipated bequest coincides with the income transfer.

$$(1 + g_n)Tr = \sum_{j=1}^{J} (1 - \psi_j)\mu_j \int a'_j(a, \varepsilon, m) d\lambda_j(a, \varepsilon, m) \tag{38}$$

7. Social security satisfies the budget constraint.

$$\sum_{j=1}^{J}\int \tau_{ss} \min\{we(\varepsilon)h, \overline{m}\} d\lambda_j(a, \varepsilon, m) = \sum_{j=1}^{J}\int b_{ss}(m) d\lambda_j(a, \varepsilon, m) \tag{39}$$

Transition path in Ino and Kobayashi (2020)

Given the fiscal policy $\tau_t = (\tau_{y,t}, \gamma_t, \chi_t, \tilde{B}_t)$, a transition path is a pair of functions $(V_t, \tilde{a}'_t, \tilde{W}_t, r_t, \mu_t, \tilde{K}_t, L_t, \tau_{c,t})_{t=0}^{T}$ which satisfies:

1. V_t is the solution to the following Bellman equation and a'_t is the associated policy function:

$$V_t(a, \epsilon) = \max_{a', c, l}\{u(c, l) + \tilde{\beta}\sum_{\epsilon'} P(\epsilon'|\epsilon)V_{t+1}(a', \epsilon')]\} \tag{40}$$

$$\text{s.t. } (1+\tau_{c,t})\, c + (1+g)a' = (1-\tau_{y,t})w_t\epsilon(1-l) + (1+(1-\tau_{y,t})r)a + \chi, \tag{42}$$
$$a' \geq \underline{a},\ l \in [0,1],\ c \geq 0,$$

where the value function at the terminal coincides with the one in the terminal stational equilibrium ($V_T(a,\epsilon) = V(a,\epsilon,\tau^{terminal})$).

2. Prices (r,w) and capital-labor input (K,L) are consistent with the firm's profit maximization;

$$\tilde{W}_t \equiv \frac{W_t}{z_t} = (1-\alpha)\left(\frac{\tilde{K}_t}{L_t}\right)^{\alpha-1} \tag{43}$$

$$r_t = \alpha\left(\frac{\tilde{K}_t}{L_t}\right)^{\alpha} - \delta. \tag{44}$$

3. Capital and Labor market clear.

$$L_t = \int \epsilon[1 - l_t(a,\epsilon)]d\mu_t(a,\epsilon), \quad \tilde{A}_t = \tilde{K}_t + \tilde{B}_t. \tag{45}$$

4. The distribution of households (46)

$$\mu_{t+1}(a',\epsilon') = \sum\sum 1\{a'_t(a,\epsilon) = a'\}P(\epsilon'|\epsilon)\mu_t(a,\epsilon), \tag{47}$$

with $\mu_0((a,\epsilon) = \mu(a,\epsilon,\tau^{ini}))$.

5. Government budget constraint satisfies.

$$\tilde{G}_t + (1+r_t)\tilde{B}_t = \tilde{B}_{t+1}(1+g) + \tau_{c,t}\tau\tilde{C}_t + \tau y_t(r_t\tilde{K}_t + \tilde{W}_t L). \tag{48}$$

Details in the numerical analysis

We discretize the continuous variable asset as follows. First, we set the upper bound of the asset a so that only a negligible fraction of households try to hold assets more than that level. The lower bound of asset $\underline{a}$ is a parameter of this model. Once we decide the upper and lower bounds, we create an evenly spaced grid of N_a points between $[\underline{a},a]$. Ino and Kobayashi (2020) use $a = 40$ and $N_a = 101$.

In order to calculate the stationary equilibrium, we used the following algorithm:

1. Make a guess on the interest rate and wage (r,w) and set (r_i,w_i).
2. i th iteration: we calculate the gap between demand and supply of assets and labor as

follow:

- Given (r_i,w_i), we solve the Bellman equation and obtain (a'_i,l). We use the backward induction when we assume households live for a finite period, and for infinitely lived households' case, we use Value Function Iteration.
- Use the policy function to obtain the distribution of households $\mu_i(a,\epsilon)$. (Details are below.)
- Once we compute prices (r_i,w_i), policy functions $a'_i(a,\epsilon)$, and household's distribution $\mu_i(a,\epsilon)$, we can compute the demand and supply of asset and labor.

3. If the demand-supply gap is small enough, then the guess is correct. Otherwise, update the guess on prices and go through the process again.

In order to compute the stationary distribution, we use the following algorithm:

- We create a finer grid space for assets with $N_a \times M$ points and interpolate the policy function on this space. Ino and Kobayashi (2020) use $M = 3$.
- If $a' \in (a_i, a_{i+1})$, we define the probability $a' = a$ as

$$p(a, z) = \frac{a_{i+1} - a'(a, z)}{a_{i+1} - a_i}. \tag{51}$$

Based on this, we can create the transition probability matrix over a' as follows: $a' = a_{i+1}$ with probability $p(a,z)$, and $a' = a_i$ with probability $1 - p(a,z)$. Then we can use the eigenvector method (Badshah, Beaumont, and Srivastava (2013)) to compute the stationary distribution.

Computation of transition path

In computing the transition path, we follow Conesa and Krueger (1999) and make a guess on (K,L).

1. Compute the initial and terminal stationary equilibrium. Let T denote the time when the economy converges to the terminal stationary equilibrium after the policy change.
2. Make a guess on a sequence of labor and capital $(L^i_t, \tilde{K}^i_t)^T_{t=0}$, $i = 0$. We can compute a sequence of prices $(r^i_t, \tilde{W}^i_t)^T_{t=0}$ from the firm' s profit maximization.
3. Given prices $(r^i_t, \tilde{W}^i_t)^T_{t=0}$, we can update $(L^{i*}_t, \tilde{K}^{i*}_t)^T_{t=0}$ as follows:
   (a) Given prices and terminal condition $V_T(a,\epsilon) = V^{SS}(a,\epsilon)$, solve the households' problem backward.
   (b) Given the policy functions, compute the distribution of households forward starting from $\mu_0(a,\epsilon) = \mu^{SS}(a,\epsilon)$.
   (c) Use the policy function and the distribution of households to update the aggregate capital and labor.
4. We check if the guess is correct or not by using the condition $\max\{|K_{t,i} - K^*_{t,i}|, |L_{t,i} - L^*_{t,i}|\} < \varepsilon$. If this condition is satisfied, done. Otherwise, update the capital and labor as follows:

$$K_{t,i+1} = \omega K_{t,i} + (1-\omega)K^*_{t,i},\ L_{t,i+1} = \omega L_{t,i} + (1-\omega)L^*_{t,i}$$

and go to step 2 again.

It is not guaranteed that this algorithm converges. We try a different value of the weight on the old guess, ω, and obtained convergence with $\omega = 0.9$.