Sustainability of Budget Deficits

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Abstract

This paper examines theoretical models that underpin studies on “sustainability of budget deficits”, which have been drawing interest in recent years, and also explains methods of empirical tests. The paper starts with a discussion on the intertemporal government budget constraint in a certainty model and then expands the discussion to under uncertainty. Under uncertainty, the issue of whether or not Ponzi schemes are feasible in a dynamically efficient economy is theoretically important. When the economy is dynamically efficient in the “strict” sense, Ponzi schemes are infeasible. Although it is possible for the risk-free interest rate to be lower than the economic growth rate on average under sufficient uncertainty, it is important to ensure that Ponzi schemes are infeasible even in such a case.

Empirical research on fiscal sustainability started with a study by Hamilton and Flavin, which sought to examine whether the present value borrowing constraint is satisfied. Later, the premise of their analysis (stationarity of primary surplus) was disputed by Wilcox and by Trehan and Walsh, and different tests were proposed. In addition, Bohn proposed another test from a different perspective. This paper explains those methods and points out their problems.

Keywords: sustainability of budget deficits, dynamic efficiency
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I. Introduction

Studies on “sustainability of budget deficits” are attracting interest in recent years. In the United States, the research started due to the budget deficit increase of the 1980s during the Reagan administration days. In Japan, the fiscal deterioration of the post-bubble era brought about the attention of this problem. However, few studies in Japan have solid theoretical models which underlies empirical studies1. In this paper, we will give a brief explanation of the theoretical models on which empirical tests are based, and also comment on the methods of empirical testing.

In section II, implication of intertemporal government budget constraint will be discussed under the economy without uncertainty. Here, it is important that feasibility of the Ponzi scheme depends on whether the interest rate is larger than the economic growth rate or not, which is related to dynamic efficiency. In section III, definitions and conditions of dynamic

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1 Hatano (2005) provides a good survey of the studies.
efficiency under certainty will be discussed, and then will be extended to the case of uncertainty. There, the proposition that the Ponzi scheme is infeasible in a dynamically efficient economy (under uncertainty) is derived. In section IV, empirical studies are summarized and some comments will be given.

II. Budget constraint of government

II-1. Intertemporal budget constraint

First we will consider the budget constraint of the government under the economy without uncertainty. For simplicity, rate of interest $r$ is supposed to be constant over time. Consider the following budget accounting identity

$$ D_{t+1} = D_t(1 + r) + G_t - T_t $$

(1)

where $D_t$ is government debt at the beginning of period $t$, $G_t$ is government expenditure (excluding interest payment), and $T_t$ is tax revenue. “Conventional” budget surplus is defined to be net increase of assets (or net decrease of debt), $-(D_{t+1} - D_t)$, which is equal to $T_t - (G_t + rD_t)$ as shown by equation (1). Primary surplus is defined to be $T_t - G_t$, the difference between tax revenue and government expenditure (excluding interest payment).

From equation (1), the following equation is obtained ($j \geq 1$).

$$ D_t = \frac{1}{(1+r)} \sum_{i=0}^{j-1} T_{t+i} - G_{t+i} + \frac{D_{t+j}}{(1+r)^j} $$

(2)

This is the intertemporal government budget constraint. Notice that this equation is satisfied whether budget deficits are sustainable or not (we have not yet clarified this meaning). However, when $j$ goes to infinity, the following condition must be held:

$$ \lim_{j \to \infty} \frac{D_{t+j}}{(1+r)^j} = 0 $$

(3)

Equation (3) is called the No-Ponzi Game (NPG) condition, which prohibits infinite borrowing by the government. When the NPG condition is satisfied, the following equation is derived.

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2 Suppose that a government bond is issued at some period and interest of debt is never repaid, then debt will grow at the rate just equal to the interest rate. In this case the NPG condition is not satisfied because $D_{t+j} / (1 + r)^j$ is constant. That is, the NPG condition is the condition that prohibits “robbing Peter to pay Paul”.

In this paper we will pay attention only to the budget constraint of narrowly defined government, and will not mention about “consolidated” government budget constraints, which includes constraint of the central bank. Based on consolidated government budget constraint, the Fiscal Theory of Price Level argues that when (narrow) government is not Ricardian, the central bank must create money to satisfy consolidated budget constraint, which determines the price level of the economy.
\[ D_t = \frac{1}{1+r} \sum_{i=0}^{\infty} \frac{T_{t+i} - g_{t+i}}{(1+r)^i} \]  

Equation (4) says that if the government debt at time \( t \) is positive, the sum of discounted present value of primary surplus from present to infinite future must be positive and equal to \( D_t \). However it is not necessary for government debt to be zero by some future date. It is sufficient that the growth rate of \( D_t \) is smaller than the interest rate.

As explained in section IV, equation (4) (or equation (3)) is used to test whether budget deficits are sustainable or not. Equation (4) is called the “present value borrowing constraint” by Hamilton and Flavin (1986). In addition, the present value borrowing constraint itself is a very loose condition at least under certainty. For example, even if debt is huge, this condition is satisfied if the growth rate of debt is smaller than the interest rate\(^3\).

Notice that the ability of government to repay may not matter when the absolute amount of debt is large if the economy also grows. In order to see this, we will divide government budget constraint as the ratio to GDP. First, let \( Y_t \) be GDP at time \( t \). Moreover, let \( d_t, g_t, \tau_t \) be \( D_t, G_t, T_t \) defined as the ratio of GDP, respectively. To simplify discussion, suppose that \( n \), the growth rate of \( Y_t \), is held constant. Dividing equation (1) by \( Y_t \), and using the relation \( Y_{t+1} = Y_t (1 + n) \), we will obtain the following equation.

\[ (1 + n)d_{t+1} = (1 + r)d_t + g_t - \tau_t \]  

(5)

From this equation, we can get

\[ d_t = \frac{1}{1+r} \sum_{i=0}^{j-1} \frac{T_{t+i} - g_{t+i}}{(1+r)^i} + \frac{d_{t+j}}{(1+y)^j} \]  

(6)

where \( 1+y \equiv (1+r)/(1+n) \). Therefore, if \( r > n \) then \( \gamma > 0 \), if \( r < n \) then \( \gamma < 0 \).

Now suppose that \( d_0 \), debt to GDP ratio at period \( t_0 \), is given. And assume that government will not repay the interest of government bonds at subsequent periods and set primary surplus always equal to zero (Ponzi scheme). When a Ponzi scheme is carried out, \( d_t \) increases at rate \( \gamma \) as equation (6) shows. Therefore, if \( r > n \), \( d_{t+j} \) increases infinitely, which leads to financial collapse of government. However, if \( r < n \), \( d_{t+j} \) approaches 0 as \( j \) increases to infinity, and government does not fail. Thus, it is dependent on the relationship between \( r \) and \( n \) whether the Ponzi scheme is feasible or not.

Now, I will return to equation (5). What amount of primary surplus is required to keep the debt GDP ratio constant? To answer this question, \( d_{t+1} = d_t = d \) is substituted into equation (5), and we can get the following equation (subscript \( t \) is omitted).

\(^3\) Let \( d \) be debt-GDP ratio, and \( r \) and \( n \) be interest rate and economic growth rate, respectively. In order for holding \( d \) constant in a steady state, primary surplus as the ratio of GDP must be equal to \((r-n)\) \( d \), and must be kept forever, as explained later. Although budget deficit is sustainable (or present value borrowing constraint is satisfied) in this case, fiscal management would be very hard when \((r-n)\) \( d \) is high.
\[ \tau - g = (r - n)d \]  

(7)

That is, in order to keep \( d \) constant, required primary surplus (as the ratio to GDP) is equal to \((r - n)\ d\). For example, if \( d \) is to be held 2.0 and \( r - n = 1\% \), this equation shows that government must produce a primary surplus equal to 2.0\% of GDP forever.

We may add that in the case of \( r < n \), equation (3) is an unnecessary condition for excluding default of government. As shown in equation (7), in order to keep \( d \) constant when \( r < n \), government can produce primary deficits. In this case, since \( d \) is constant, \( D_{t+j} \) increases at the rate of economic growth \( n \). But, since \( r < n \), \( D_{t+j} / (1 + r)^j \) increases infinitely as \( j \) goes to large, which violates the NPG condition (equation (3)). However, since \( d \) is constant, there are no financial problems. But, such a policy cannot be carried out without limit since government debt crowds out private investment, which increases interest rate as a result of reduced capital. If sufficient debt is issued and sufficient private capital is crowded out, then \( r > n \) will be realized at some future date.

Now, consider the case of \( r > n \) once again. Equation (6) helps us to calculate the necessary amount of primary surplus in order to reach some certain level of debt \( j \) period ahead. Consider the case of reducing debt GDP ratio by \( \Delta \) at \( j \) periods ahead. Substituting \( d_{t+j} = d_t - \Delta \) into equation (6), and assume that primary surplus (as the ratio to GDP) is held constant over time, we can get the following equation.

\[ \tau - g = (r - n)d_t + (r - n)\frac{\Delta}{(1 + r)^j - 1} \]  

(8)

The first term of the RHS of this equation is the necessary surplus to keep the current debt level, and the second term represents additional requirements for reducing debt to the target level. The second term becomes large, as \( \Delta \) (the gap between current and target debt) becomes large, and as \( j \) (the length of time) becomes short\(^4\).

The long term debt outstanding of central and local government in Japan was 196\% of GDP at the end of fiscal year 2012. So, we assume \( d = 2.0 \) and calculate the value of the RHS of equation (8) for different values of \( \Delta \) and \( j \) (\( r = 0.02 \) and \( n = 0.01 \) are also assumed). The column of \( \Delta = 0 \) of Table 1 is the case where \( d = 2.0 \) is maintained forever. In that case, it is just equal to \((r - n)\ d\), the first term of the RHS of equation (8). This means that necessary

\(^4\ r > n \) is assumed. Equation (8) is related to the index called sustainability gap indicator which was employed by the EU committee (The Long-term Sustainability of Public Finance in the European Union, Oct. 2006). This indicator consists of three terms. The first is IBP (Initial Budgetary Position), required change in budgetary balance in order to keep the initial debt GDP ratio. The second is DR (Debt Requirement in 2050), a necessary change in budgetary balance in order to reduce the debt GDP ratio to 0.60, the target value by 2050. And the third is LTC (Long Term Changes in the Primary Balance), an additional change in budgetary position in order to cope with the increase in expenditure by population aging by 2050. As easily understood, the gap between the first term of equation (8) and the current primary surplus corresponds to IBP. And the gap between the second term of equation (8) and the current primary surplus corresponds to DR. If “implicit debt” such as future payment of public pension and medical insurance is included, LTC can be calculated by using equation (8).
primary surplus is 2.0% of GDP (this amount is needed forever). Although the required primary surplus does not differ so much when $\Delta$ is small, Table 1 shows a heavy burden is needed when we want to reduce debt by half ($\Delta = 1.0$). Of course, it is unrealistic to realize this in 10 years. But in the case of $j = 30$, required primary surplus for 30 years is 5% of GDP (Since $d = 1.0$ after target reduction is achieved, 1% of primary surplus is needed forever in order to keep this level of $d$).

**II-2. Domar’s proposition**

There is a famous proposition by Domar, which characterize the relationship between deficit and debt. Domar (1944) showed that the continuing budget deficit does not necessarily lead to default of government when the economy grows. The budget deficit in this context is a conventional one (the gap between government expenditure including interest payment and tax revenue), not a primary deficit. As is often confused, Domar’s proposition always holds if the growth rate of the economy is positive, irrelevant to relative magnitude between interest rate and economic growth rate. Now, I will summarize Domar’s proposition.

Domar showed that debt GDP ratio ($d$) converges to a certain finite value when the growth rate of the economy is positive, and government does not fail, if budget deficit remains to be constant relative to GDP. This proposition is easily derived. From equation (5), we can get

$$d_{t+j} = \sum_{i=1}^{j} \frac{\delta_{t+j-i}}{(1+n)^j} + \frac{d_t}{(1+n)^j}$$

(9)

where $\delta_t = rd_t + g_t - \tau_t$. Domar considered the case where $\delta_t$ is a constant. By substituting $\delta_t = \delta$ into equation (9), and by using the formula for the sum of the geometric series, it can be easily shown $n$ must be positive in order for $d_t$ to converge to a finite value. In this case, the following equation is derived from equation (9).

$$\lim_{j \to \infty} d_{t+j} = \frac{\delta}{n}$$

That is, when the growth rate of the economy is positive, debt GDP ratio will converge to
\[\delta / n,\text{ and the government will never fail if the government can keep deficit to a constant relative to GDP. Moreover, it is also important that the convergent value of } d \text{ is independent from the initial position. For example, when } n = 1\% \text{ and } \delta = 1\%, \text{ debt outstanding will be } 100\% \text{ of GDP in the long run. If } \delta = 2\% \text{ (} n = 1\%), \text{ debt approaches } 200\% \text{ of GDP in the long run (even in this case government does not fail). Debt GDP ratio becomes low as the growth rate of the economy becomes high.}

However, even if public deficit is kept constant relative to GDP, fiscal management is not so easy. For example, suppose that \( r = 2\% \) and \( n = 1\% \). If \( \delta = 2\% \), \( d \) will be 2.0 in the long run. But in this case, interest-payment expense is 4\% of GDP \((r \cdot d)\). Therefore, in order to keep \( \delta \) constant (2\%), necessary primary surplus is positive and 2\% of GDP. This shows that Domar’s proposition does not guarantee the easiness of fiscal management.

### III. Dynamic efficiency

**III-1. Dynamic efficiency under certainty**

As we have explained in II.-1, in the economy under certainty, the path of \( d_t \) depends on \( r \) and \( n \). The relationship between \( r \) and \( n \) is important not only when considering government budget constraint. It is important because it is related to a “dynamic efficiency”. In this section, conditions of a dynamic efficiency and inefficiency are derived under the economy without uncertainty. First, definitions of a dynamic efficiency and inefficiency will be given.

**Definition 1:** The economy is said to be dynamically inefficient if total consumption at some period can be increased without reducing total consumption at any other period. The economy is said to be dynamically efficient if total consumption at some period cannot be increased without reducing total consumption at any other period.

This definition focuses on total consumption at each period. Although there is another definition focusing on utility of each generation in an overlapping generations model, I will use Definition 1 for a while.

In order to simplify the discussion below, we will use the Solow growth model. The growth rate of labor force is constant and equal to \( n \), and output is produced with capital and labor, with constant returns technology. Output per worker is therefore written as \( y_t = f(k_t) \), where \( y_t \) is output per worker, and \( k_t \) is capital labor ratio at period \( t \). We also assume usual property of production function: that is, \( f'(k) > 0, f''(k) < 0 \). The path of capital labor ratio obeys the following equation.

\[
k_{t+1} = \frac{1}{1+n}[k_t(1-\delta) + f(k_t) - c_t]
\]

(10)

where \( \delta \) is depreciation rate, and \( c_t \) is consumption per worker at period \( t \). Under some conditions, \( k_t \) approaches to some stationary value. In what follows, I will denote the value of
each variable in a steady state by dropping the subscript \( t \).

In order to obtain the steady state value of \( k \), \( k_{t+1} = k_t = k \) is substituted into equation (10), and \( (n + \delta)k = f(k) - c \) is obtained. The steady state value of \( k \) is the solution of this equation. Therefore, steady state consumption is equal to \( c = f(k) - (n + \delta)k \), and the condition for maximizing \( c \) is easily derived as \( f'(k) = n + \delta \). Since the interest rate is determined to be equal to \( f'(k) - \delta \) in a competitive economy, the condition for maximizing \( c \) is equivalent to \( r = n \). This condition is well known as the Golden Rule. We denote \( k^* \) as (steady state) capital labor ratio in the Golden Rule. When \( k < k^* \) (under accumulation), then \( r > n \) holds. When \( k > k^* \) (over accumulation), then \( r < n \) holds.

In the economy under certainty, \( r > n \) is the condition for dynamic efficiency, and \( r < n \) is the condition for dynamic inefficiency. We will confirm this according to the definition of dynamic efficiency/inefficiency. Assume that the path of consumption per worker and capital labor ratio from period \( t \) to the future is given. Then suppose that consumption per worker at period \( t \) is increased by one unit, but consumption (per worker) at a later period is kept as before. If such a scheme is feasible, the economy is said to be dynamically inefficient, and if infeasible, the economy is said to be dynamically efficient.

Equation (10) shows that a one unit increase in consumption at time \( t \) decreases next period capital \( k_{t+1} \) by \( 1 / (1 + n) \) unit. Then, we will denote change in \( k_{t+1} \) by \( dk_{t+1} = (-1 / (1 + n) < 0) \). Under decreased capital, production and consumption at time \( t+1 \) is conducted. But since consumption is assumed to be kept as before, \( k_{t+2} \) must be decreased. Change in \( k_{t+2} \) is given by

\[
dk_{t+2} = \frac{1 - \delta + f'(k_{t+1})}{1 + n} dk_{t+1}
\]

where \( dk_{t+2} \) is change in \( k_{t+2} \), \( f'(k_{t+1}) \) is the marginal product of capital evaluated at the initial path of capital labor ratio. Change in \( k_{t+j} \), \( dk_{t+j} \), can be derived by a similar manner.

\[
dk_{t+j} = \frac{1 - \delta + f'(k_{t+j-1})}{1 + n} dk_{t+j-1} = \left( \prod_{i=1}^{j-1} \frac{1 - \delta + f'(k_{t+i})}{1 + n} \right) dk_{t+1}
\]

Future consumption can be kept as before if \( dk_{t+j} \) approaches 0 as \( j \) goes to infinity. Therefore, it is sufficient for dynamic inefficiency that the coefficient of \( dk_{t+1} \) in equation (11) approaches 0 when \( j \) goes to infinity. Since \( f'(k) - \delta = r \) holds in a competitive economy, this condition is equivalent to

\[
\lim_{j \to \infty} \prod_{i=1}^{j} \frac{1 + r_{t+i}}{1 + n} = 0
\]

which, in steady state, is equivalent to \( r < n \). In this way, it was shown that \( r < n \) is the condition for dynamic inefficiency.

Then, what happens when \( \prod_{i=1}^{j} (1 + r_{t+i}) / (1 + n) \) grows without bound as \( j \) goes to infinity? In this case, \( k \) will be decreased to 0 by some future period, when output drops to 0 and the initial level of consumption cannot be maintained anymore. That is, the economy is
dynamically efficient. Under steady state, since the condition $\lim_{x \to \infty} \prod_{i=1}^{x} [(1 + r_{t+i})/(1 + n)] = \infty$ is equivalent to $r > n$. So, this is the condition for dynamic efficiency.

As mentioned above, the condition for dynamic efficiency/inefficiency depends on whether the growth rate of the economy is larger than the interest rate. We also mentioned that the NPG condition is not necessary for the sustainability of government budget when $r < n$. The NPG condition, or present value borrowing constraint matters only when $r > n$ holds, or the economy is dynamically efficient. It is well known that there are some cases that $r < n$ holds in the economy of optimizing consumers, such as the OLG model and finite horizon model (Diamond (1965) and Blanchard and Fischer (1989)).

III-2. Model under uncertainty

Although there exists only one interest rate in the economy under certainty if we ignore the difference of liquidity, different assets have different interest rates in the actual economy according to the risk they have. Since the only rate of interest does not exist, the condition derived under certainty cannot be applied directly. In this section, discussion will be extended to the uncertain economy, where the state of nature is stochastic, and people cannot foresee the realized value beforehand, but only know the probability distribution of the state. First, we must clarify the definition of dynamic efficiency and inefficiency under uncertain economy. The definition often employed is as follows.

**Definition 2:** If total consumption at some period (at some realized value of the state) can be increased without decreasing total consumption at any period and at any state of nature, then the economy is said to be dynamically inefficient. If it is impossible, the economy is said to be dynamically efficient.

Using this definition, Blanchard and Weil (1992) discussed whether the Ponzi scheme is feasible or not (the Ponzi scheme is a policy that government roll over debt forever without ever levying taxes). And they found that in some cases the Ponzi scheme can be feasible in uncertain economy. On the other hand, Abel, Mankiw, Summers and Zeckhauser (1989) and Ball, Elmendorf and Mankiw (1998) employed another definition of dynamic efficiency and argued that the Ponzi scheme is infeasible. The following is the definition they used.

**Definition 3:** If utility of some generation can be increased (at some realized value of the state) without decreasing utility of any other generation at any state, the economy is said to be dynamically inefficient. If this is impossible, the economy is said to be dynamically efficient.

The definition here defines the dynamic efficiency by utility of each generation, Definition 2 by total consumption at each period, in which case Pareto improvement can be possible by reallocation of consumption between different generations without changing total
consumption at each period. Hence Definition 3 is a “strong” definition in a sense that there is no room for Pareto improvement by such reallocation. Using Definition 3, Ball et al. showed that the Ponzi scheme is infeasible in the dynamically efficient economy. In what follows we will first summarize the discussion of Ball et al., and then summarize the discussion of Blanchard et al.

Notice that it can be possible in some realization of state that government can reallocate (ex post) so that utility of some generations are increased without reducing utility of any other generations, even if the economy is dynamically efficient. because dynamic efficiency requires the impossibility of such reallocation at any state and it is not contradictory to succeed in some state. Then, Ball et al. discussed not the Ponzi scheme but the Ponzi gamble, the policy to roll over government debt unless debt does not exceed some certain upper limit. Once debt reaches some upper limit, the Ponzi gamble is over. We say that it is successful if debt can be rolled over forever, and that it is failed when debt reaches some upper limit. In what follows, the upper limit of debt is assumed to some constant fraction of total wage payment at each period.

When the government launches a Ponzi gamble, the ratio of debt to total wage payment is increasing according to the ratio $Q_t = \prod_{i=1}^{t} [(1 + r_i)/(1 + g_i)]$, where $r_i$ is realized interest rate at period $i$, and $g_i$ is realized growth rate of wage. If $Q_t$ increases without bound, debt will exceed some fraction of total wage, and the Ponzi gamble will fail. Hence, $Q_t$ needs to be bounded so that the Ponzi gamble succeeds even if the economy is dynamically efficient.

Now, consider the reallocation of consumption as below. To simplify the discussion, the 2 period OLG model is assumed here, and population of each generation is assumed to be equal to 1. At period 0, the young worker gives up $\delta W_0$ of consumption, and equal amount of consumption of the old is increased, where $\delta$ is some small number. At period 1, the same kind of within period reallocation of consumption is conducted, except that transfer at this period is $(1 + r_1)$ times transfer at period 0. In the same way, reallocation at period $t$ is assumed $(1 + r_t)$ times reallocation at period $t - 1$. Such transfer at time $t$ is equal to $[\prod_{i=1}^{t} (1 + r_i)] \delta W_0$. Hence the ratio of transfer to total wage is equal to $\delta Q_t$. When the Ponzi gamble is successful, $Q_t$ must be bounded as we explained before. Since $\delta$ is assumed to be small, $\delta Q_t$ is also small so that such reallocation of consumption is feasible. Moreover, since total consumption is held constant, path of capital stock is unaffected, and wage and interest rate are also unaffected.

We will show that such a transfer is Pareto improving. First, the utility of the old at period 0 increases. The consumption of the young at period $t$ ($t = 0,1,2,\ldots$) is decreased, but their consumption of next period is increased. Total change in utility of generation $t$ (generation born at time $t$) is unchanged if their initial path of consumption is optimized. Therefore, in

\footnote{Utility of generation $t$ (born at period $t$) is given by $u(c_t^y) + E_t v(c_{t+1}^o)$, where $c_t^y$ is consumption of the young at period $t$, $c_{t+1}^o$ is consumption of the old at period $t + 1$, and $E_t$ is the expectation conditional on information available at period $t$. The first order condition for utility maximization is $u'(c_t^y) = E_t (1 + r_{t+1}) v'(c_{t+1}^o)$, where $r_{t+1}$ is the interest rate at period $t + 1$. The change in utility from such reallocation of consumption is given by $-Q_t \delta W_t [u'(c_t^y) - E_t (1 + r_{t+1}) v'(c_{t+1}^o)]$, which is equal to zero if the initial path of consumption satisfies the Euler equation.}
some cases the Ponzi gamble is successful even if the economy is dynamically efficient. That is Pareto improving reallocation of consumption is (ex post) possible. Although the above argument is based on direct control of resource allocation, Ball et al. showed Pareto improvement is also possible using tax and transfer under decentralized economy.

Now, the Ponzi scheme is a policy in which the Ponzi gamble succeeds in any realization of state. And when the Ponzi gamble succeeds, Pareto improving reallocation of consumption is possible. If the Ponzi scheme is feasible in the dynamically efficient economy, then Pareto improving reallocation is possible in any realization of state, but is contradictory to the definition of dynamic efficiency. Therefore, the following proposition is derived.

**Proposition 1:** If the economy is dynamically efficient, then the Ponzi scheme is not feasible.

Abel et al. (1989) derived conditions for dynamic efficiency in the economy under uncertainty using the same definition of dynamic efficiency, which states that sufficient condition for dynamic efficiency is that the rate of return of market portfolio is larger than the growth rate of the economy at any periods and in any realization of state. They also showed this is equivalent to cash outflow from the firm that exceeds the cash inflow to the firm (investment) at all periods and in all states. They also investigated several OECD countries and found no evidence of dynamic inefficiency.

According to the empirical study of Abel et al., the Japanese economy is not dynamically inefficient. And according to the proposition of Ball et al., the Ponzi scheme is infeasible in a dynamically efficient economy. In the paper of Ball et al., they compared the interest rate of government debt and the growth rate of GDP, and found the mean value of the former is smaller than that of the latter historically. But this is consistent with dynamic efficiency. The purpose of their study is focused on the probability that the Ponzi gamble succeeds in a dynamically efficient economy. They estimated the random process of the gap between interest rate and GDP growth rate, based on which they studied the probability of success and failure of the Ponzi gamble. They pointed out that such a policy is not wise even if the probability of success is high, and also undesirable because of crowding out of capital.

On the other hand, Blanchard and Weil (1992) examined whether the Ponzi scheme is feasible or not in a 2 period OLG model, employing a “weak” definition of a dynamic efficiency. They considered two cases; (1) the only source of uncertainty is technological shock, (2) storage technology is subject to random shock. In case (1), they showed that even if the economy is dynamically efficient, mean interest rate of government bonds can be lower than the mean growth rate of the economy, in which case the Ponzi scheme is shown to be infeasible. In case (2), they showed the Ponzi scheme is feasible even if the economy is dynamically efficient. But in this case there exists a market failure of risk sharing between the old and the young, which is thought to be the cause of their proposition (if they do not employ the “weak” definition of dynamic efficiency, there would be no Pareto improving reallocation in the dynamically efficient economy). So, I will not discuss case (2). And case
(1) will be discussed in Appendix B, where comparison of interest rate and growth rate (in mean value) is shown to be misleading.

III-3. Risk free interest rate and economic growth rate

In this section it is shown that risk free interest rate can be lower than economic growth rate in the dynamically efficient economy. The following discussions are the same as Abel et al. (1989).

For simplicity, an infinite horizon model is assumed instead of an OLG model. Utility of a representative individual is given by

$$E_t \sum_{i=0}^{\infty} \beta^i u(C_{t+i})$$

where $C_t$ is the consumption at period $t$, $\beta$ is subjective discount factor ($0 < \beta < 1$), and $E_t$ is the expectation operator conditional on the information available at period $t$. Output is produced by “fruit tree”, the only factor of production, and obeys following stochastic processes:

$$Y_t = Y_{t-1}(1 + g_t)$$

where $Y_t$ is output at period $t$, and $g_t$ is the growth rate of the economy, and assumed to be random variables, independently and identically distributed with $t$, satisfying $1 + g_t > 0$ for all $t$. Here, $1 + g_t$ is assumed to follow log normal distribution, with mean $1 + g$. If $\ln(1 + g_t) \sim N(\mu, \sigma^2)$, then $E(1 + g_t) = \exp(\mu + \sigma^2/2)$ holds. That is, $1 + g = \exp(\mu + \sigma^2/2)$ holds (the detail of the derivation is in the Appendix A).

Rate of return, $R_{t+1}$, of any assets in this economy must satisfy the following Euler equation.

$$E_t \left[ \beta^u(C_{t+1}) u'(C_t) \right] R_{t+1} = 1$$

This equation must hold for $R^f$, the rate of return of riskless asset. We also assume that output cannot be stored, so the goods market equilibrium condition is given by $C_t = Y_t$. If utility function is specified as $u(C) = \ln C$, we can get

$$R^f_{t+1} = \frac{1}{\beta E_t[(1+g_t)^{-1}]} = \beta^{-1} \exp(\mu - \sigma^2/2) = \frac{\beta^{-1}(1+g)}{\exp(\sigma^2)}$$

Equation (15) shows that risk free interest rate, $R^f_{t+1}$, can be lower than mean economic growth rate, $1 + g$, if $\sigma^2$ is sufficiently large. In addition, sufficient condition for dynamic efficiency is that $Y_t > 0$ holds at every period in any states. Therefore, dynamic efficiency is

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6 See Appendix A for the derivation of second equality in equation(15).
guaranteed from the assumption: $1 + g_t > 0$ holds for all $t$. If the consumer is more risk averse than the logarithmic utility case, we can show that risk free rate of interest is smaller than the one predicted by equation (15).

**IV. Empirical studies of fiscal sustainability**

There are several methods for testing the sustainability of budget deficits, some of which are applied to the Japanese economy.

**IV-I. Hamilton and Flavin**

Hamilton and Flavin (1986) tested whether the present value borrowing constraint was satisfied. Intertemporal government budget constrain is given by

$$D_t = \frac{1}{1+r} \sum_{i=0}^{\infty} \frac{T_{t+i} - G_{t+i}}{(1+r)^i} + \frac{D_{t+j}}{(1+r)^j}$$

They investigated whether the second term of RHS of the above equation approaches zero when $j$ goes to infinity. If the NPG condition is satisfied, the second term of RHS is rewritten as

$$\lim_{j \to \infty} \frac{D_{t+j}}{(1+r)^j} = (1 + r)^t \left[ \lim_{T \to \infty} \frac{D_T}{(1+r)^T} \right] = 0$$

Considering the uncertainty in the economy, government budget constraint can be rewritten as expected value (conditional on the information at period $t$)

$$D_t = E_t \left[ \sum_{i=0}^{\infty} \frac{T_{t+i} - G_{t+i}}{(1+r)^{i+1}} \right] + A_0 (1 + r)^t$$

where $A_0 = \lim_{T \to \infty} E_t [D_T/(1+r)^T]$, the expected value of future debt discounted at time 0. Hamilton and Flavin tested whether $A_0 = 0$ is satisfied, in particular they did the following three tests.

1. Testing whether $D_t$ is a stationary process or not. ($D_t$ is stationary if $A_0 = 0$)
2. Testing $A_0 = 0$ using reduced form of equation (16).
3. Testing $A_0 = 0$ using simultaneous equation of (16).

The first procedure is as follows. If $A_0 = 0$ is satisfied, the RHS of equation (16) contains only the first term, which follows stationary process. However, if $A_0 > 0$, since the second term of equation (16) grows at the rate $(1 + r)^t$, $D_t$ follows nonstationary process. Therefore, in order to investigate whether $A_0$ is 0 or not, it is necessary and sufficient to test whether $D_t$ is stationary or not. They tested the hypothesis that $D_n$ has a unit root and found the null hypothesis was rejected. They also tested whether the process of primary surplus has a unit
root, and found no evidence of nonstationarity. Therefore, they found no evidence that budget deficit is not sustainable.

The second procedure is the test of $A_0 = 0$ by estimating equation (16) directly. However, since the first term of RHS of the equation cannot be observed, future primary surplus, $T_{t+i} - G_{t+i}$, is assumed to be predicted by stochastic process of the past. If this assumption is satisfied, equation (16) can be written as follows,

$$D_t = a + A_0 (1 + r)^t + b_1 D_{t-1} + \cdots + b_p D_{t-p} + c_0 S_t + \cdots + c_{p-1} S_{t-p+1} + \varepsilon_t$$

where $S_t$ is the primary surplus at period $t$. Notice that the lagged value of $D_t$ is included in the above equation, which is necessary to deal with serial correlation of the error term. Testing $A_0 = 0$ is just testing the coefficient of $(1 + r)^t$ is equal to 0. They included $(1 + r)^t$ as an explanatory variable using $r$ as 1.12%, the average value of interest rates from 1960 to 84, and estimated equation by OLS, and found hypothesis $A_0 = 0$ cannot be rejected.

The third is a test based on simultaneous equation. In this method, stochastic process of $S_t$, and equation (16) is estimated as simultaneous equations, then $A_0 = 0$ is tested. Also in this case, the hypothesis that $A_0$ is zero is not rejected.

Notice that their study is based on U.S. data from 1960 to 84. There may have been a structural change in and after 1981 when the Reagan Administration took office, as they mentioned in their paper. However, they could not find structural changes because the degree of freedom was not sufficient. This is a problem we always confront. When the budget deficit becomes serious, and the economist is urged to test the sustainability of budget deficits, such kinds of econometric studies will offer no clear evidence because the degree of freedom is not sufficient.

IV-2. Nonstationarity

Wilcox (1989) insisted that Hamilton and Flavin’s test is incorrect, taking as an example that primary surplus and debt follows nonstationary process but discounted present value of future debt converges to 0 (that is, present value borrowing constraint is satisfied). The example he pointed out is as follows.

Assume that the government set primary surplus at each period just equal to one half of interest payment of the same period (for the discussion below to be valid, it is not necessary that primary surplus is equal to one half of interest payment, but sufficient that it is between 0 and 1). In this case, debt grows according to $D_{t+1} = D_t (1 + r / 2)$, which is nonstationary. Primary surplus is also nonstationary because it is proportional to debt. However, since the discounted value of debt obeys the equation $D_{t+j} / (1 + r)^j = D_t \left[ (1 + \frac{r}{2})^j / (1 + r)^j \right]$, which converges to 0 as $j$ goes to infinity, that is, the present value borrowing constraint is satisfied. Therefore, for testing sustainability of budget deficits, it is incorrect only to test whether debt is nonstationary or not.
Wilcox also discussed the problems arising from the assumption of a constant interest rate, and specification of present value borrowing constraint. As for the former assumption, Wilcox used a realized real interest rate. As for the latter assumption, additional explanation may be necessary.

First, let $S_t$ be $(T_t - G_t) / (1 + r)^t$, the primary surplus at period $t$ discounted at time 0. And let $D'_t$ be $D_t / (1 + r)^t$, then intertemporal government budget constraint is written as follows.

$$D'_t = E_t \sum_{i=0}^{\infty} S_{t+i} + A_t$$

where $A_t$ represents the discrepancy of debt at time $t$ from the present value borrowing constraint, which satisfies $A_t = \lim_{t \to \infty} E_t D'_t$. Hamilton and Flavin assumed that $A_t$ is always equal to $A_0$, which is correct, Wilcox argued, if debt is stationary process. But when debt is nonstationary, Hamilton and Flavin’s procedure is problematic, and $A_t$ becomes stochastic which satisfies $E_t A_{t+1} = A_t$. Wilcox argued that in this case the correct procedure is to test whether $A_t$ follows stationary process around mean 0, not to test $A_t = 0$. According to this argument, he estimated the process of discounted value of debt and found its mean value does not converge to 0. That is, his result did not support the sustainability of budget deficits.

Trehan and Walsh (1991) avoided the problems arising from nonstationarity by another method. They showed that even if primary surplus is nonstationary, the necessary and sufficient condition for satisfying present value borrowing constraint is that there exists cointegration between debt and primary surplus (for this proposition to be valid, the expected value of interest rate must be constant over time). They also showed that sufficient condition is that the first order difference of debt, which is equal to conventional budget deficit, is stationary process, provided that the lower bound of interest rate is positive. By estimating the stochastic process of debt, they found that the first difference of debt is stationary. That is, they showed that present value borrowing constraint was satisfied.

Derivation of the proposition of Trehan and Walsh is slightly technical. Moreover, since it is the second method they actually carried out, we will skip the first proposition. The second proposition that government does not fail when the first difference of debt follows stationary process is considered to be an extension of Domar’s proposition. I will explain this briefly. First, consider the discounted value of debt at $j$ periods ahead.

$$E_t \left[ \prod_{i=0}^{j-1} (1 + r_{t+i}) \right]^{-1} D_{t+j}$$

Conventional budget deficit, the gap between government expenditure including interest payment and tax revenue, is equal to net increase of government debt, $\Delta D_t$. Therefore, if the first difference of debt is stationary, stochastic process of debt is represented as

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7 Wilcox also found that the unit root test of Hamilton and Flavin is not robust with estimation periods. While Hamilton and Flavin showed debt and primary surplus are stationary, once the first year observation (year 1962) is excluded, Wilcox found that their results were not supported.
\( \Delta D_t = \delta + A(L)e_t. \) Where \( A(L) \) represents lag polynomial, \( A(L) = \sum_{i=0}^{\infty} a_i L^i, \) \( L \) is the lag operator, \( e_t \) is white noise, and \( \delta \) is some constant. In order for \( \Delta D_t \) to be stationary, \( \sum_{i=0}^{\infty} |a_i|^2 < \infty \) must be held. And it is also assumed that \( r_t > \rho (> 0) \) is satisfied for all \( t \). From the equation \( D_{t+j} = D_t + \sum_{i=1}^{j} \Delta D_{t+i} \) and \( \Delta D_t = \delta + A(L)e_t, \) the following equation is derived.

\[
\lim_{j \to \infty} E_t D_{t+j} = D_t + \lim_{j \to \infty} j\delta
\]

Since \( r_t \geq \rho \) is satisfied for all \( t \), we have

\[
\lim_{j \to \infty} E_t \left[ \prod_{i=0}^{j-1} (1 + \rho_{t+i}) \right]^{-1} D_{t+j} \leq \lim_{j \to \infty} E_t \frac{D_{t+j}}{(1+\rho)^j} = \lim_{j \to \infty} \frac{D_{t+j} + j\delta}{(1+\rho)^j} = 0
\]

Therefore, expected discounted value of debt at \( j \) periods ahead approaches 0 when \( j \) goes to infinity. This is because the numerator, debt, grows in linear order, while the denominator grows at an exponential rate when conventional deficit is stationary process. In this way, it was shown that the sufficient condition for present value borrowing constraint to be satisfied was that conventional budget deficit (including interest payment) is a stationary process.

**IV-3. Government response**

If the government always responds to increase in debt by increasing budget surplus, the risk of financial collapse of government will be low. Bohn (1998) assumed that the government responds to increase in debt according to the following equation.

\[
s_t = \rho dr_t + \alpha Z_t + \epsilon_t \equiv \rho dr_t + \mu_t
\]

where \( s_t \) is primary surplus as ratio to GDP and \( d_t \) is debt GDP ratio, \( Z_t \) is other factors (vector) that affect government response, \( \epsilon_t \) is a disturbance term, and \( \alpha \) and \( \rho \) are parameters to be estimated. If \( \rho > 0 \) is observed, then the government (actually) acts so as to increase surplus when debt is increased.

According to Bohn, there are no positive correlations between primary surplus and debt in the U.S. when both series are simply plotted. But he found a positive correlation when other factor \( Z_t \) was included in the regression equation. He had the tax smoothing model of Barro in mind and as \( Z_t \) he used the variables that reflected temporal increase in government expenditure (due to war and others), and business cycle factor. He found positive \( \rho \), which was between 0.028 and 0.054, and concluded U.S. budget deficit was sustainable.

Bohn pointed out in another paper that stochastic discount factor, marginal rate of substitution between consumption today and tomorrow, is theoretically appropriate for discounting future debt and surplus rather than interest rate of debt (mean value or realized value at each period). Hamilton and Flavin employed a mean interest rate, and Wilcox employed a realized interest rate. Bohn’s method is free from this criticism because estimation requires no assumption about interest rate.
Finally, let me show how debt GDP ratio evolves when the government responds to debt increase according to the above equation. As shown in II-1, the path of debt follows $(1 + n)d_{t+1} = (1 + r)d_t - s_t$. Substituting the government response equation into this equation, we can get

$$d_{t+1} = \frac{1+r-\rho}{1+n} d_t - \frac{1}{1+n} \mu_t$$

Therefore, $d_{t+j}$ is given by

$$d_{t+j} = \left(\frac{1+r-\rho}{1+n}\right)^j d_t - \frac{1}{1+n} \sum_{i=1}^{j} \left(\frac{1+r-\rho}{1+n}\right)^{i-1} \mu_{t+j-i}$$

If we ignore the second term of the RHS of the equation (since $\mu$ represents business cycle factors, it is reasonable to ignore this term in the long run),

$$\frac{D_{t+j}}{(1+r)^j} = \frac{d_{t+j}(1+n)^j Y_t}{(1+r)^j} = \left(\frac{1 + r - \rho}{1 + r}\right)^j d_t Y_t$$

is derived. This equation shows present value borrowing constraint is satisfied only if $\rho > 0$ is satisfied.

**IV-4. Caveats**

Finally we will mention some caveats on empirical studies. First, definition of debt is not clear cut. Theoretically, implicit debt, such as social security benefits in the future, should be included in debt, as Kotlikoff (1992) pointed out. But few empirical studies took it into consideration. Secondly, such econometric studies only observe the past trend of debt (and deficit). For example, when debt was found to grow at a rate equal to the interest rate, what can be said is only deficit will not be sustainable if government pursues the same policy as before. This criticism is also applicable to Bohn’s study.

The third is related to what happens when the government faces an actual budget crises. In such a case, a plunge in the price of the government bonds and a sudden rise of interest rates will arise. Economists will need long and sufficient periods in order to find out budget crises by the statistical procedure discussed in this paper, which needs sufficient degrees of freedom. This makes us quite doubtful about the significance of the empirical study of the kind discussed in this paper.

**V. Conclusion**

In this paper, we pointed out that dynamic efficiency is theoretically important in relation to the sustainability of budget deficits. In the uncertain world, it is possible that interest rate is lower than the growth rate of the economy (in mean value) even in the dynamically efficient economy. In this paper, we referred to the important proposition that the Ponzi scheme is
infeasible in such an economy. Of course, the Ponzi “gamble” can be successful ex post in such an economy. However, considering the risk and the effect of crowding out capital, we pointed out that the Ponzi gamble is by no means wise policy. We also summarize the empirical methods for testing sustainability of budget deficits, emphasizing the relationship with a theoretical model.

As shown in this paper, the (necessary) condition of budget sustainability in the economy under certain world conditions is very weak. Even if debt is large, if future generations can pay the burden, budget is said to be sustainable (or present value borrowing constraint is satisfied). The studies of generational accounting use this condition to predict the extra burden of future generations, which is used for policy evaluation consistent with a lifecycle model. There are many undesirable effects of public debt (including implicit debt in social security programs), one of which is crowding out of capital, and another is the generational imbalance of tax burden. In an actual economy, these problems will become serious before budget deficit is unsustainable. Therefore, unsustainability may have a second importance.

References


Appendix A. Moment of a lognormal distribution

Let \( x \) be a random variable which follows lognormal distribution, that is, \( \ln x \sim N(\mu, \sigma^2) \). Expected value of \( x \) and \( x^{-1} \) is given by

\[
Ex = \exp(\mu + \sigma^2/2) \tag{A.1}
\]

\[
Ex^{-1} = \exp(-\mu + (\alpha\sigma)^2/2) \tag{A.2}
\]

More generally, the following result is obtained.

\[
Ex^{\alpha} = \exp(\alpha\mu + \sigma^2/2) \tag{A.3}
\]

From equation (A.1) and (A.3), the following result is derived.

\[
\text{Var } x = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1] \tag{A.4}
\]

(Proof)

Let \( y = \ln x \), then \( y \) is a random variable which follows normal distribution with mean \( \mu \) and variance \( \sigma^2 \). Let \( g(y) \) and \( f(x) \) be density function for \( y \) and \( x \), respectively. Since \( g(y) \) is density function of normal distribution, it is given by

\[
g(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(y - \mu)^2}{2\sigma^2} \right]
\]

On the other hand, since \( y \) and \( x \) are a monotonic relationship, \( f(x) \) is derived from the formula for transformation of variable.

\[
f(x) = g(y) \frac{dy}{dx} = \frac{1}{\sqrt{2\pi\sigma^2}x} \exp \left[ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right]
\]

From this equation, we can get the following equation.

\[
Ex = \int_0^\infty xf(x)dx = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right]dx
\]

Let \( y \) represent \( \ln x \), and by using relation: \( dx = \exp(y) \, dy \), the above equation is rewritten as

\[
Ex = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{1}{2} \left( \frac{y-\mu}{\sigma} \right)^2 \right] \exp(y) \, dy \tag{a.1}
\]

Moreover, the following equation holds.

\[
-\frac{1}{2} \left( \frac{y-\mu}{\sigma} \right)^2 + y = -\frac{1}{2} \left( \frac{y-\mu}{\sigma} \right)^2 + \left( \mu + \frac{\sigma^2}{2} \right) \tag{a.2}
\]

where \( \mu' = \mu + \sigma^2 \). If we substitute equation (a.2) into equation (a.1), we can get the expected value of \( x \).
As for expected value of $x^\alpha$,

$$Ex = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{1}{2} \left( \frac{y - \mu}{\sigma} \right)^2 \right] \exp \left( \mu + \frac{\sigma^2}{2} \right) dy = \exp \left( \mu + \frac{\sigma^2}{2} \right)$$

Let $y = \ln x$, and using the relation $x^\alpha = \exp (\alpha y)$, the following equation is derived,

$$Ex^\alpha = \int_{0}^{\infty} x^\alpha f(x) dx = \int_{0}^{\infty} \frac{x^\alpha}{\sqrt{2\pi\sigma^2}} \frac{1}{x} \exp \left[-\frac{1}{2} \left( \frac{\ln x - \mu}{\sigma} \right)^2 \right] dx$$

$$= \exp \left( \alpha \mu + \frac{1}{2} (\alpha \sigma)^2 \right)$$

where $\mu'' = \mu + \alpha \sigma^2$. In this way, we can get equation (A.3). Notice that equation (A.2) is the special case for $\alpha = -1$ in equation (A.3). Finally, equation (A.4) is derived from $\text{Var } x = Ex^2 - (Ex)^2$. 
Appendix B. Blanchard and Weil model

Blanchard and Weil (1992) examined the feasibility of the Ponzi scheme in a dynamically efficient economy by using a stochastic general equilibrium model. They showed that it is misleading to argue based on the average value of the interest rate of safety asset (government debt) and economic growth rate. They used a 2 period OLG model and considered cases where the only source of uncertainty is (1) technological shock which affects production function, and (2) shock in storage technology. Among these, the latter is considered to be the case where there exists market failure, the failure of risk sharing between the old and the young at the same point of time, which is considered to make the Ponzi scheme feasible. That is, it is the “weak” definition of dynamic efficiency which enables Pareto improving reallocation of resources. So, I take up only the case of (1) here. It is shown that the mean value of interest rate of safety asset can become lower than the mean value of economic growth rate in a dynamically efficient economy. But, the Ponzi scheme is infeasible even in this case.

First, assume that expected utility of generation \( t \), the generation born at period \( t \), is given by,

\[
(1 - \beta) \ln C_{1,t} + \beta E_t \ln C_{2,t+1}
\]  

(B.1)

where \( C_{1,t} \) is consumption of the young at period \( t \), and \( C_{2,t+1} \) is consumption of the old at period \( t + 1 \). Each generation supplies one unit of labor when young, and get wage \( W_t \), and choose consumption and saving. Saving in the first period is used as capital input for production in the second period. The (gross) return from capital is used consumption in the second period. The budget constraint at each period is given by

\[
C_{1,t} + K_{t+1} = W_t
\]

\[
C_{2,t+1} = R_{t+1} K_{t+1}
\]

where \( K_{t+1} \) is the saving in period \( t \) and it is equal to capital at next period, and \( R_{t+1} \) represents gross rate of return from capital.

Population of each generation is assumed to be 1, and production function is given by the following equation

\[
Y_t = \Gamma_t K_t^\alpha
\]

(B.2)

where \( \Gamma_t \) is a random variable that captures technological shocks. Here, the log of \( \Gamma_t \) is assumed to follow i.i.d. normal distribution with mean \( \mu \) and variance \( \sigma^2 \). \( \alpha \) is capital share, a constant which satisfies \( 0 < \alpha < 1 \). In addition, capital is assumed to depreciate completely in one period when used as input for production. Output and factor markets are assumed to be competitive. Therefore \( W_t \) and \( R_t \) are determined to be equal to marginal product of capital and labor, respectively. Finally, after random variable \( \Gamma_t \) realized, the return of capital and
labor is assumed to be determined, and based on these realized returns, consumption and
saving is assumed to be determined.

Wage and return from capital is determined as
\[ W_t = (1 - \alpha) \Gamma_t K_t^\alpha, \]
\[ R_t = \alpha \Gamma_t K_t^{\alpha-1}. \]
Using these equations and the conditions of consumer optimization, the path of consumption and
capital can be derived.

\[
K_{t+1} = (1 - \alpha) \beta \Gamma_t K_t^\alpha \\
C_{1,t} = (1 - \alpha)(1 - \beta) \Gamma_t K_t^\alpha \\
C_{2,t} = \alpha \Gamma_t K_t^\alpha 
\]  \hspace{1cm} (B.3)

Notice that since \( K_{t+1} \) is determined after \( \Gamma_t \) is realized, the only uncertainty generation \( t \) faces
is uncertainty in \( R_{t+1} \) (which reflects \( \Gamma_{t+1} \)).

Next, the above equation is rewritten in logarithmic form. Let \( k_t, y_t, \) and \( \nu_t \) be the log of
\( K_t, Y_t, \) and \( \Gamma_t, \) respectively. From the assumption of the probability distribution of \( \Gamma_t, \) \( \nu_t \)
follows the i.i.d. normal distribution. Production function is as follows.

\[ y_t = \alpha k_t + \nu_t \]  \hspace{1cm} (B.4)

Let \( w_t \) and \( r_t \) be the log \( W_t \) and \( R_t. \) These are given by the following.

\[ w_t = \ln(1 - \alpha) + \alpha k_t + \nu_t \]
\[ r_t = \ln \alpha + (\alpha - 1) k_t + \nu_t \]

Moreover, from equation (B.3), the transition equation of capital is given by

\[ k_{t+1} = \ln[(1 - \alpha) \beta] + \alpha k_t + \nu_t \]  \hspace{1cm} (B.5)

which shows \( k_t \) follows AR (1) process. And this shows that \( y_t \) also follows AR (1) process.

According to Blanchard and Weil, the necessary and sufficient condition for dynamic
efficiency in a “weak” sense is that the expected value of \( r_t (= \ln R_t) \) is non-negative. Therefore,
from equation (B.4), necessary and sufficient condition for dynamic efficiency is shown to be

\[ \mathbb{E} r_t = \ln \alpha + (\alpha - 1) \mathbb{E} k_t = \ln \left[ \frac{\alpha}{(1 - \alpha) \beta} \right] \equiv \ln(1 + \theta) \geq 0 \]

where expectation operator \( \mathbb{E} \) represents unconditional expectation. Notice that the above
condition is equivalent to \( \theta \geq 0. \) That is, \( \theta \geq 0 \) is the condition for dynamic efficiency.\(^8\)

\(^8\) \( 1 + \theta \equiv \frac{\alpha}{(1 - \alpha) \beta} \) is the ratio of capital income share \( (\alpha) \) to saving rate \( ((1 - \alpha) \beta) \). As is well
known, the Golden Rule is realized in the neoclassical growth model when share of capital income is
equal to the saving rate. If the saving rate is higher than capital income share, then the economy is
overaccumulated and dynamically inefficient \( (\theta < 0) \). If the saving rate is lower than capital income
share, then the economy is dynamically efficient \( (\theta > 0) \).
Next, the determination of interest rate of safety asset will be explained. Assume that government bond is safety asset. Let \( R^f \) be the gross rate of return of the government bond. From the consumer optimization, the following Euler equation must be satisfied.

\[
E^t \left[ \frac{\beta C_{t+1}^{-1}}{(1-\beta)C_{t+1}^{-1}} R^f_{t+1} \right] = 1
\]

From equation (B.6) and (B.3),

\[
R^f_{t+1} = \frac{(1-\beta)}{\beta} \frac{C_{t+1}^{-1}}{E^t C_{t+1}^{-1}} = \frac{\alpha K_{t+1}^{-1}}{E^t R^f_{t+1}}
\]

is derived. Since \( \ln \Gamma \) follows normal distribution with mean 0 and variance \( \sigma^2 \), \( E[\Gamma^{-1}] = \exp(\sigma^2/2) \) holds. (see Appendix A). Therefore, risk free interest rate is determined as \( R^f_{t+1} = \exp(-\sigma^2/2) \alpha K_{t+1}^{-1} \). If we take the logarithm, this equation becomes

\[
r^f_{t+1} = \ln R^f_{t+1} = \ln \alpha + (\alpha - 1)k_{t+1} - \sigma^2/2
\]

Although a risk-free interest rate is a deterministic variable at period \( t \), since \( k_t \) is a stochastic variable which follows AR (1) process, \( r^f_t \) also follows AR (1) process. By using the equation of \( r_t \) and \( r^f_t \), we can get the relation between \( r^f_t \) and \( r_t \).

\[
r^f_{t+1} = r_{t+1} - v_{t+1} - \sigma^2/2
\]

That is, the rate of return of safety asset is lower as variance of technological shock is larger. Also, expected value and variance of return of safety asset can be calculated as follows\(^9\).

\[
E^t r^f_{t+1} = \ln(1 + \theta) - \sigma^2/2
\]

\[
\text{Var} r^f_{t+1} = (1 - \alpha)^2 \text{Var} k = \frac{(1-\alpha)^2}{1-\alpha^2} \sigma^2 = \frac{1-\alpha}{1+\alpha} \sigma^2
\]

Finally, we will discuss whether the Ponzi scheme is feasible. Let \( D_t \) be government debt at period \( t \). Assume that initial debt \( D_0 \) is given and that maturity of debt is one year and debt is refinanced at each period and never financed by tax increase. Then, debt GDP ratio at period \( t \) is given by

\[
\frac{D_t}{Y_t} = \left( R^f_t R^f_{t-1} \cdots R^f_1 \right) \frac{D_0}{Y_t} = \left[ \frac{R^f_t}{Y_t/Y_{t-1}} \right] \left[ \frac{R^f_{t-1}}{Y_{t-1}/Y_{t-2}} \right] \cdots \left[ \frac{R^f_1}{Y_1/Y_0} \right] \frac{D_0}{Y_0}
\]

The above equation can be written in logarithmic form,

\[
d_t - y_t = d_0 - y_0 + \sum_{s=1}^{t} \left( r^f_s - (y_s - y_{s-1}) \right)
\]

\(^9\) Variance of \( k \) is calculated using the property that \( k \) follows AR (1) process as shown in equation (B.5).
where \( d_t = \ln D_t \). By using equation (B.4) and (B.5), the second term of RHS of equation (B.8) can be calculated as follows.

\[
r_t^f - (y_t - y_{t-1}) = \ln(1 + \theta) - \sigma^2/2 - v_t \tag{B.9}
\]

Therefore, it was shown that the log of debt GDP ratio follows the process of random walk with drift. If we substitute equation (B.9) into equation (B.8), we can get

\[
d_t - y_t = d_0 - y_0 + t \cdot [\ln(1 + \theta) - \sigma^2/2] - \sum_{s=1}^{t} v_s
\]

Therefore, expected value and variance of \( d_t - y_t \) are given by the following equation.

\[
E(d_t - y_t) = d_0 - y_0 + t \cdot [\ln(1 + \theta) - \sigma^2/2] \\
\text{Var}(d_t - y_t) = t \cdot \sigma^2
\]

Notice that variance of \( (d_t - y_t) \) is increasing with \( t \). Therefore, the path of future \( (d_t - y_t) \) should not be evaluated by only expected value, since it is nonstationary. In addition, if \( \sigma^2 \) is sufficiently large and return of safety asset is smaller than expected value of economic growth rate, expected value of \( (d_t - y_t) \) goes to minus infinity as \( t \) grows. However, the expected value of \( D_t / Y_t \) does not converge to 0. This can be shown by the formula for expected value of lognormal distribution.

\[
E[D_t / Y_t] = \exp(d_0 - y_0) \exp[t \cdot (\ln(1 + \theta) - \sigma^2/2)] = (D_0 / Y_0)(1 + \theta)^t
\]

This equation shows that expected value debt GDP ratio diverges in the dynamically efficient economy \( (\theta > 0) \), even if interest rates of government bonds are lower than the economic growth rate.

When the degree of risk aversion is higher than a logarithmic utility case, the rate of interest of a riskless asset still becomes lower. Blanchard and Weil also considered the case of the following utility function.

\[
(1 - \beta) \ln C_{1,t} + \beta \ln [E_t C_{2,t+1}]^{1/(1-\gamma)} \tag{B.12}
\]

where \( \gamma \) is the degree of relative risk aversion. Notice that \( \gamma = 1 \) is the case equivalent to equation (B.1). Risk-free interest rate in this case is derived in the similar way as before. Let \( R_{t+1}^{f,\gamma} \) be the gross interest rate of safety asset when the degree of relative risk aversion is given by \( \gamma \). Then the following Euler equation must be satisfied.

\[
\frac{1 - \beta}{C_{1,t}} = \beta R_{t+1}^{f,\gamma} \frac{E_t C_{2,t+1}^{1-\gamma}}{E_t C_{2,t+1}} \tag{B.12}
\]
The above equation can be solved for $R_{t+1}^{f,y}$.

$$R_{t+1}^{f,y} = \frac{E_t R_{t+1}^{-1}}{E_t \Gamma_{t+1}^{-y}} a K_{t+1}^{a-1} = \frac{\exp((1 - \gamma)^2 \sigma^2/2)}{\exp[y^2 \sigma^2/2]} a K_{t+1}^{a-1} = \exp[(1 - 2\gamma)\sigma^2/2] a K_{t+1}^{a-1}$$

If we take the logarithm of this equation,

$$r_{t+1}^{f,y} = r_t - \nu_t + \sigma^2 (1 - 2\gamma)/2$$

is derived. When $\gamma > 1$ is satisfied, it is shown that $r_{t+1}^{f,y} > r_{t+1}^{f,1}$ holds and that $r_{t+1}^{f,y}$ will be lower as $\gamma$ is larger. In order to derive debt GDP ratio in this case, we first derive the equation corresponding to (B.9).

$$r_{t+1}^{f,y} - (y_t - y_{t-1}) = \ln(1 + \theta) + \sigma^2(1 - 2\gamma)/2 - \nu_t$$

By using this result, the log of debt GDP ratio can be calculated as follows.

$$d_t - y_t = d_0 - y_0 + t \cdot [\ln(1 + \theta) + \sigma^2(1 - 2\gamma)/2] - \sum_{s=1}^t \nu_s$$

Expected value of $D_t / Y_t$ is derived by using the formula for lognormal distribution.

$$\mathbb{E}[D_t/Y_t] = [D_0/Y_0](1 + \theta)^t \exp[(1 - \gamma)\sigma^2 t]$$

In this case, expected value of debt GDP ratio will converge to 0 if $\gamma$ is sufficiently large. However, since $d_t - y_t$ is nonstationary, the probability that $d_t - y_t$ exceeds some certain level will not be 0. We must be careful that sustainability of budget deficits cannot be evaluated by only expected value.

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10 By substituting $C_{2,t+1} = R_{t+1} K_{t+1}$ into utility function, the following equation is derived,

$$(1 - \beta) \ln C_{t,t} + \frac{\beta}{1 - \gamma} \ln E_t R_{t+1}^{-1} + \beta \ln K_{t+1}$$

which shows that the allocation between $C_{1,t}$ and $K_{t+1}$ are independent from $\gamma$. Therefore, $C_{1,t} = (1 - \beta) W_t$, $K_{t+1} = \beta W_t$ holds, which is just equivalent to the equation derived under the case of $\gamma = 1$ (see equation (B.3)). By using these equations and formula of lognormal distribution (see Appendix A.), Equation (B.12) is derived.